Local chiral NN potentials and light-nuclei structure

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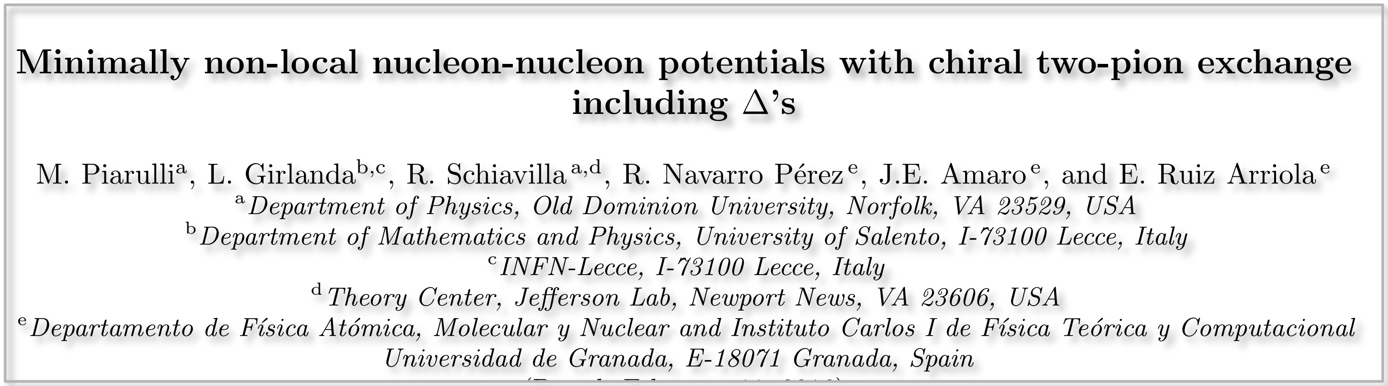

Minimally non-local nucleon-nucleon potentials with chiral two-pion exchange including Δ's

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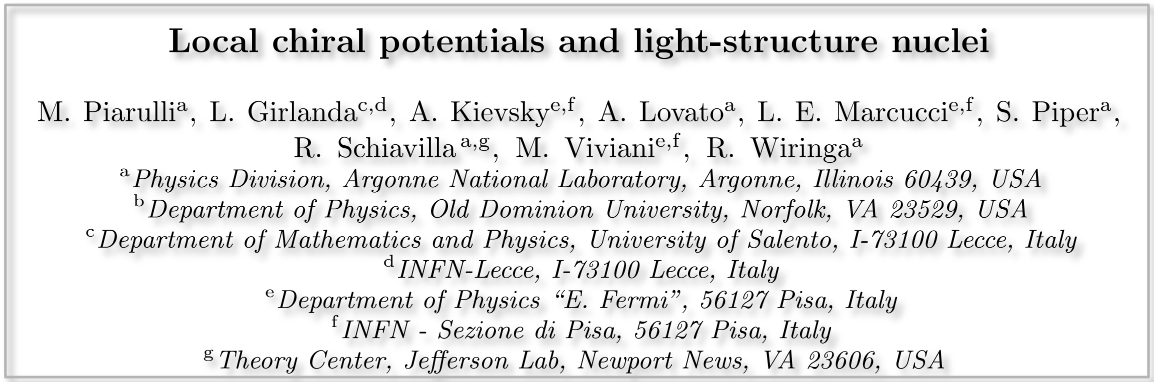
in preparation



Local chiral potentials and light-structure nuclei

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❖ The derivation of nuclear forces from χEFT has been a topic of active interest for the past 25 years

Previous work:

- S. Weinberg, Phys. Lett. B **251**, 288 (1990); Nucl. Phys. B **363**, 3 (1991)
- C. Ordóñez, L. Ray, and U. van Kolck, Phys. Rev. Lett. **72**, 1982 (1994); Phys. Rev. C **53**, 2086 (1996)
- N. Kaiser, R. Brockmann, and W. Weise, Nucl. Phys. A **625**, 758 (1997)
- N. Kaiser, S. Gerstendörfer, and W. Weise, Nucl. Phys. A **637**, 395 (1998)
- N. Kaiser, Phys. Rev. C **61**, 014003 (1999)
- N. Kaiser, Phys. Rev. C **62**, 024001 (2000)
- E. Epelbaum, W. Glöckle, and U.-G. Meißner, Nucl. Phys. A **637**, 107 (1998); **671**, 295 (2000)
- N. Kaiser, Phys. Rev. C **63**, 044010 (2001); **64**, 057001 (2001); **65**, 017001 (2001)
- D. R. Entem and R. Machleidt, Phys. Rev. C **66**, 014002 (2002); **68**, 041001 (2003)
- E. Epelbaum, W. Glöckle, and U.-G. Meißner, Eur. Phys. J. A **19**, 125 (2004)
- E. Epelbaum, W. Glöckle, and U.-G. Meißner, Nucl. Phys. A **747**, 362 (2005)
- H. Krebs, E. Epelbaum, and Ulf.-G. Meißner, Eur. Phys. J. A **32**, 127 (2007)

Recent work:

- A. Ekström, G. Baardsen, C. Forssen, G. Hagen, M. Hjorth-Jensen, G. R. Jansen, R. Machleidt, W. Nazarewicz, T. Papenbrock, J. Sarich, and S. M. Wild, Phys. Rev. Lett. **110**, 192502 (2013)
- A. Gezerlis, I. Tews, E. Epelbaum, S. Gandolfi, K. Hebeler, A. Nogga, and A. Schwenk, Phys. Rev. Lett. **111**, 032501 (2013)
- A. Gezerlis, I. Tews, E. Epelbaum, M. Freunek, S. Gandolfi, K. Hebeler, A. Nogga, and A. Schwenk, Phys. Rev. C **90**, 054323 (2014)
- D. R. Entem, N. Kaiser, R. Machleidt, and Y. Nosyk, Phys. Rev. C **91**, 014002 (2015); **92**, 064001 (2015)
- N. Kaiser, Phys. Rev. C **92**, 024002 (2015)
- E. Epelbaum, H. Krebs, and U.-G. Meißner Phys. Rev. Lett. **115**, 122301 (2015)

General Consideration:

WHY?

- ❖ Many of the available versions of chiral potentials are strongly non-local
  - ➢ Non-localities due to contact interactions
  - ➢ Non-localities due to regulator functions

⇨ $\mathbf{p} \rightarrow -i\boldsymbol{\nabla}$ relative momentum operator

- ❖ Non-local potentials hard to handle, for example in Quantum Monte Carlo (QMC) calculations

GOAL:

- ❖ Construct a local χEFT NN potential with chiral TPE including Δ-isobar:
  - ➢ Minimize the number of non-localities due to contact interactions and remove those due to the regulator functions

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Minimally nonlocal nucleon-nucleon potentials with chiral two-pion exchange including Δ's

  - ➢ Set to zero the LECs multiplying these non-localities

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Local chiral NN potentials and light- nuclei structure

Construction of minimal non-local potentials:

$$\boxed{v_{12} = v_{12}^{\mathrm{EM}} + v_{12}^{\mathrm{L}} + v_{12}^{\mathrm{S}}}$$

v_{12}^{EM} : EM interaction component

- ➢ Leading Coulomb interaction
- ➢ Second order Coulomb interaction
- ➢ Darwin-Foldy interaction
- ➢ Vacuum polarization interaction
- ➢ Magnetic moment interaction

pp system

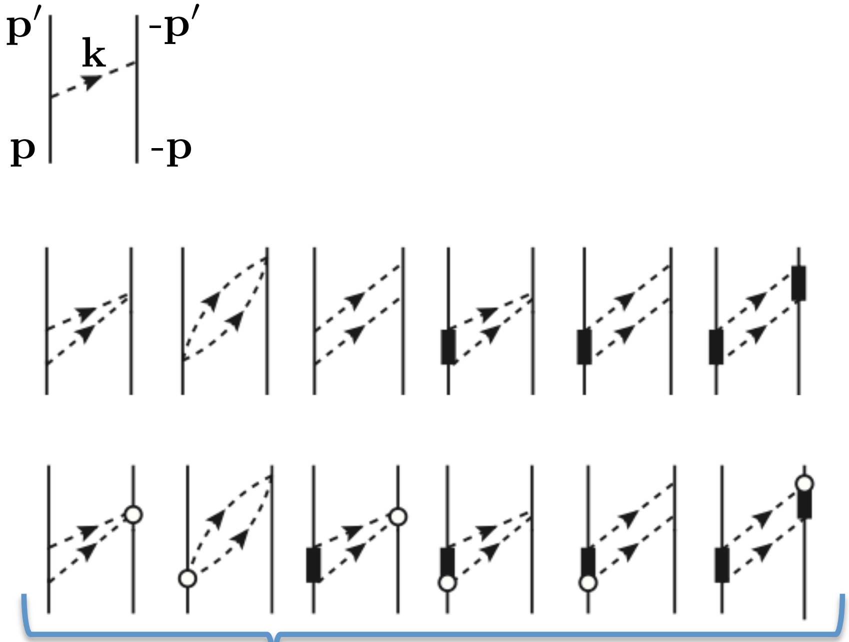
np and nn systems

v_{12}^{L} : long-range component

LO : Q^0

NLO : Q^2

N2LO : Q^3



- ➢ $\mathbf{k} = \mathbf{p}' - \mathbf{p}$
- ➢ Dependence on g_A, F_π and $h_A = 3\,g_A/\sqrt{2}$

$c_1,\ c_2,\ c_3,\ c_4\ (\mathcal{L}_{\pi N}^{(2)})$
$b_3 + b_8\ (\mathcal{L}_{\pi N\Delta}^{(2)})$

taken from π-N scattering

H. Krebs, E. Epelbaum, U.-G. Meissner (2007)

Coordinate-space $v_{12}^{\rm L}$:

$$v_{12}^{\rm L} = \left[\sum_{l=1}^{6} v_{\rm L}^{l}(r)\, O_{12}^{l}\right] + v_{\rm L}^{\sigma T}(r)\, O_{12}^{\sigma T} + v_{\rm L}^{tT}(r)\, O_{12}^{tT}$$

- $O_{12}^{l=1,\dots,6} = [\mathbf{1}\,,\ \boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2\,,\ S_{12}] \otimes [\mathbf{1}\,,\ \boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2]$

$$S_{12} = 3\,\boldsymbol{\sigma}_1\cdot\hat{\mathbf{r}}\,\boldsymbol{\sigma}_2\cdot\hat{\mathbf{r}} - \boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2$$

- $O_{12}^{\sigma T} = \boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2\, T_{12}$
- $O_{12}^{tT} = S_{12} T_{12}$

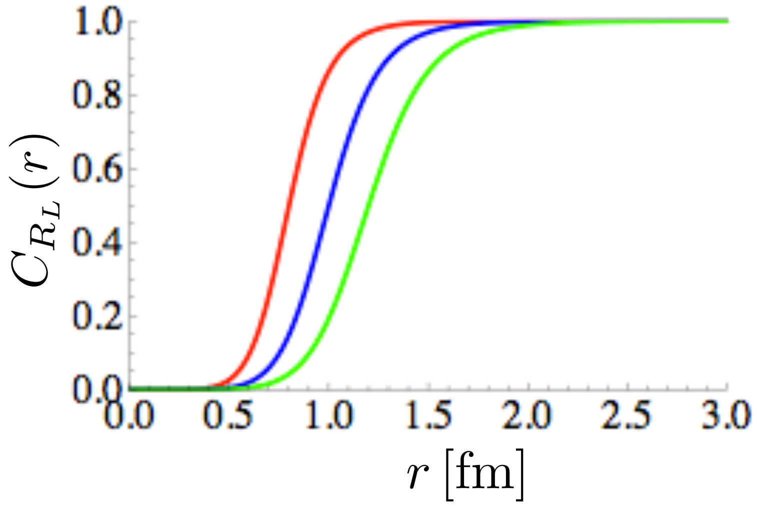
$$T_{12} = 3\,\tau_{1z}\tau_{2z} - \boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2$$

CD terms

❖ $v_L^l(r)\,, v_L^{\sigma T}(r)\,, v_L^{tT}(r)$ ⇨ divergencies of type $1/r^n\ , 1 \le n \le 6$

$$C_{R_{\rm L}}(r) = 1 - \frac{1}{(r/R_{\rm L})^6\, e^{(r-R_{\rm L})/a_L} + 1}$$

- $R_L = (0.8, 1.0, 1.2)$ fm
- $a_L = R_L/2$



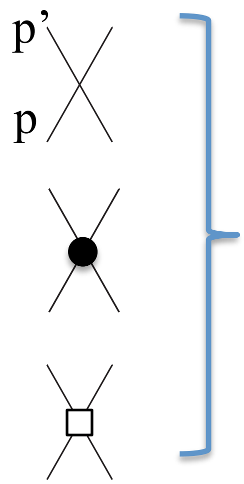
Momentum-space v_{12}^{S}:

$$\boxed{v_{12}^{\mathrm{S}}(\mathbf{k},\mathbf{K}) = v_{12}^{\mathrm{S,CI}}(\mathbf{k},\mathbf{K}) + v_{12}^{\mathrm{S,CD}}(\mathbf{k},\mathbf{K})}$$

LO : Q^0

NLO : Q^2

N3LO : Q^4



34=24 CI+ 10 CD LECs to fix

- ➢ $\mathbf{K} = \frac{1}{2}(\mathbf{p}+\mathbf{p}')$
- ➢ $\mathbf{k} = \mathbf{p}' - \mathbf{p}$

- ❖ In the NLO and N3LO contact interactions terms proportional to K^2 and K^4 have been removed by Fierz rearrangements: $P^{\,\mathrm{exc}}\,|f\rangle = -\,|f\rangle$ $\langle f|O|i\rangle = -\langle f|P^{\,\mathrm{exc}}\,O|i\rangle$

$$P^{\,\mathrm{exc}} = \frac{1+\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2}{2}\,\frac{1+\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2}{2}\,P^{\,\mathrm{space}} \rightarrow \mathbf{k}\rightarrow -2\,\mathbf{K} \text{ and } \mathbf{K}\rightarrow -1/2\,\mathbf{k}$$

Example:

$$K^m \longrightarrow -\frac{1+\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2}{2}\,\frac{1+\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2}{2}\,\frac{k^m}{2^m}$$

with m=2 or 4

- ❖ Of course mixed terms as $k^2\,K^2$ or $\mathbf{K}\times\mathbf{k}$ can not Fierz-transformed away
  - ➢ $k^2\,K^2 \longrightarrow -\frac{1+\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2}{2}\,\frac{1+\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2}{2}\,K^2\,k^2$

Coordinate-space v_{12}^{S}:

$$\boxed{v_{12}^{\mathrm{S}} = \left[\sum_{l=1}^{19} v_{\mathrm{S}}^{l}(r)\, O_{12}^{l}\right] + \cancel{\left\{ v_{\mathrm{S}}^{p}(r) + v_{\mathrm{S}}^{p\sigma}(r)\, \boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2 + v_{\mathrm{S}}^{pt}(r)\, S_{12} + v_{\mathrm{S}}^{pt\tau}(r)\, S_{12}\, \boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2 \;,\; \mathbf{p}^2 \right\}}}$$

- ➤ $O_{12}^{l=1,\ldots,6} = [\mathbf{1}\,,\,\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2\,,\,S_{12}] \otimes [\mathbf{1}\,,\,\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2]$
- ➤ $O_{12}^{l=7,\ldots,11} = \mathbf{L}\cdot\mathbf{S}\,,\,\mathbf{L}\cdot\mathbf{S}\,\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2\,,\,(\mathbf{L}\cdot\mathbf{S})^2\,,\,\mathbf{L}^2\,,\,\mathbf{L}^2\,\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2$
- ➤ $O_{12}^{l=12,\ldots,19} = [\mathbf{1}\,,\,\underbrace{\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2\,,\,S_{12}\,,\,\mathbf{L}\cdot\mathbf{S}}]\otimes[T_{12}\,,\,\underbrace{\tau_1^z+\tau_2^z}]$

not well contraint

❖ For the short-range terms the regularization is achieved by employing a local regulator

$$\boxed{\widetilde{C}_{R_{\mathrm{S}}}(k) = e^{-R_{\mathrm{S}}^2 k^2/4} \longrightarrow C_{R_{\mathrm{S}}}(r) = \frac{1}{\pi^{3/2} R_{\mathrm{S}}^3} e^{-(r/R_{\mathrm{S}})^2}}$$

❖ In combination with $R_L = (0.8, 1.0, 1.2)$ fm ➡ $R_S = (0.6, 0.7, 0.8)$ fm

➡ $\Lambda_{\mathrm{S}} = 2/R_{\mathrm{S}}\ \ (700, 600, 500)$ MeV

Fitting Procedure I:

- In this work the LECs are fixed by fitting the pp and np Granada database up to laboratory frame energies $E_{\mathrm{lab}} = 125$ MeV and $E_{\mathrm{lab}} = 200$ MeV, the deuteron binding energy and the nn scattering length
    - $3\,\sigma$-criterion to remove inconsistencies in the database [1]
    - There are 2493 exp data up to 125MeV (3476 data up to 200 MeV)
    - There are N sets each one corresponding to a different experiment
    - Each data set contains measurements at fixed energy and different scattering angle (except total cross sections)

- We fit first phase shifts and then refine the fit by minimizing the χ^2 obtained from a direct comparison with the database

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[1] R. Navarro Pérez, J.E. Amaro, and E. Ruiz Arriola, Phys. Rev. C **88**, 064002 (2013)
http://www.ugr.es/ amaro/nndatabase/

Fitting Procedure II:

- The total figure of merit is defined as the usual χ^2 function $$\boxed{\chi^2 = \sum_{t=1}^{N} \chi_t^2}$$
- An experiment may have a specified systematic error (normalized data), no systematic error (absolute data), or an arbitrarily large systematic error (floated data)
- In all cases the χ_t^2 for a data set is given by:

$$\chi_t^2 = \sum_{i=1}^{n} \frac{(o_i/Z_t - t_i)^2}{(\delta o_i/Z_t)^2} + \frac{(1 - 1/Z_t)^2}{(\delta_{\text{sys}}/Z_t)^2}$$

  - o_i and t_i are the measured and calculated values of the observable at point i
  - δo_i and δ_{sys} are the statistical and systematic errors
  - Z_t is a scaling factor chosen to minimize the χ_t^2

$$Z_t = \left(\sum_i^n \frac{o_i t_i}{\delta o_i^2} + \frac{1}{\delta_{\text{sys}}^2} \right) \Bigg/ \left(\sum_i^n \frac{t_i^2}{\delta o_i^2} + \frac{1}{\delta_{\text{sys}}^2} \right)$$

Fitting Procedure III:

- To minimize the total χ^2, we use the Practical Optimization Using No Derivatives (for Squares), POUNDerS

M. Kortelainen *et al.* Phys. Rev. C **82**, 024313 (2010)

- $\text{Model a}:(R_L, R_S) = (1.2, 0.8)$

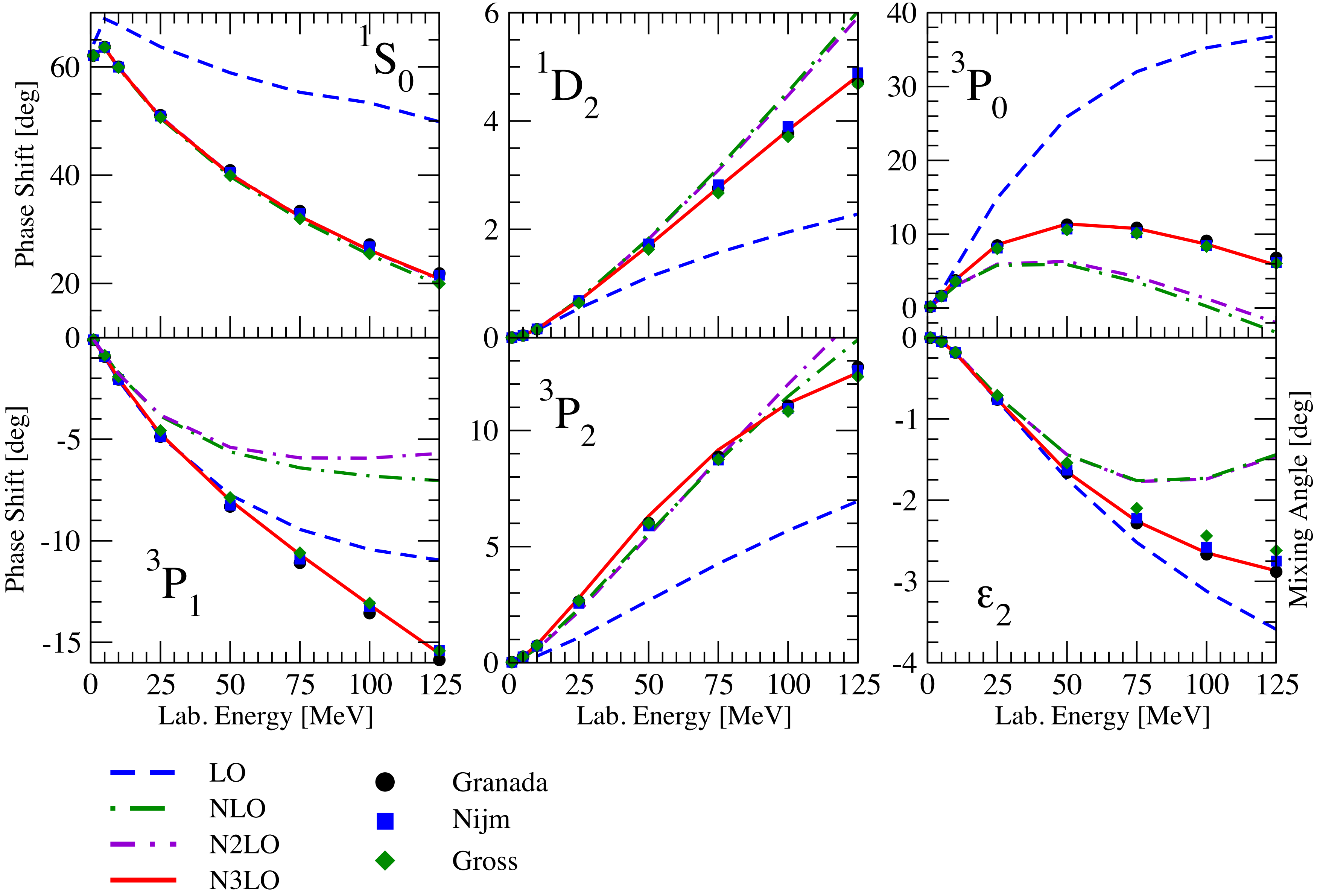
  $\text{Model b}:(R_L, R_S) = (1.0, 0.7)$

  $\text{Model c}:(R_L, R_S) = (0.8, 0.6)$

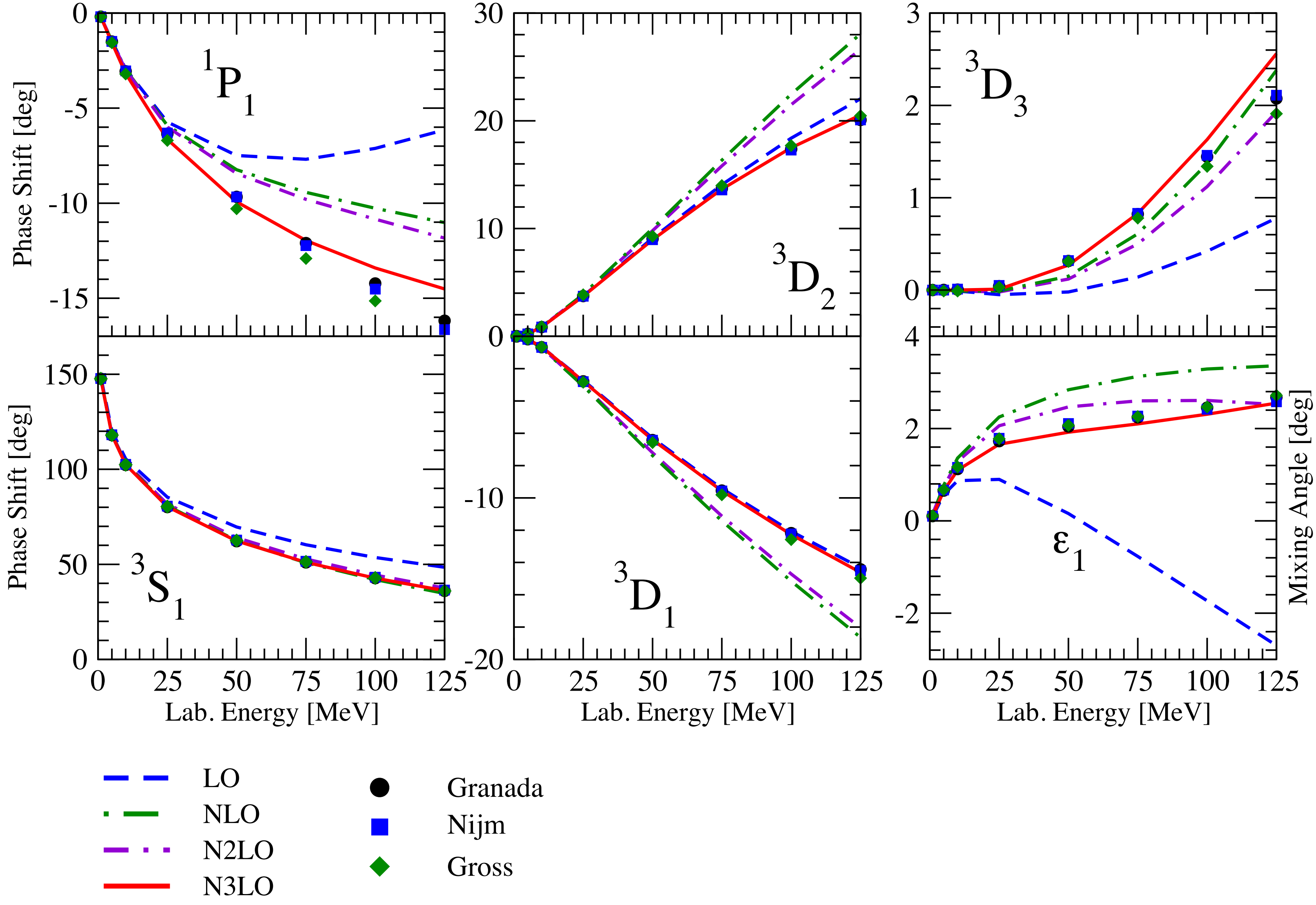
model	order	$R_{\rm L}$ (fm)	$R_{\rm S}$ (fm)	$E_{\rm LAB}$ (MeV)	χ^2/datum
Model b	LO	1.0	0.7	125	59.88
Model b	NLO	1.0	0.7	125	2.18
Model b	N2LO	1.0	0.7	125	2.32
Model b	N3LO	1.0	0.7	125	1.07
Model a	N3LO	1.2	0.8	125	1.05
Model c	N3LO	0.8	0.6	125	1.11
Model a	N3LO	1.2	0.8	200	1.37
Model b	N3LO	1.0	0.7	200	1.37
Model c	N3LO	0.8	0.6	200	1.40

Table 1: Potential models.

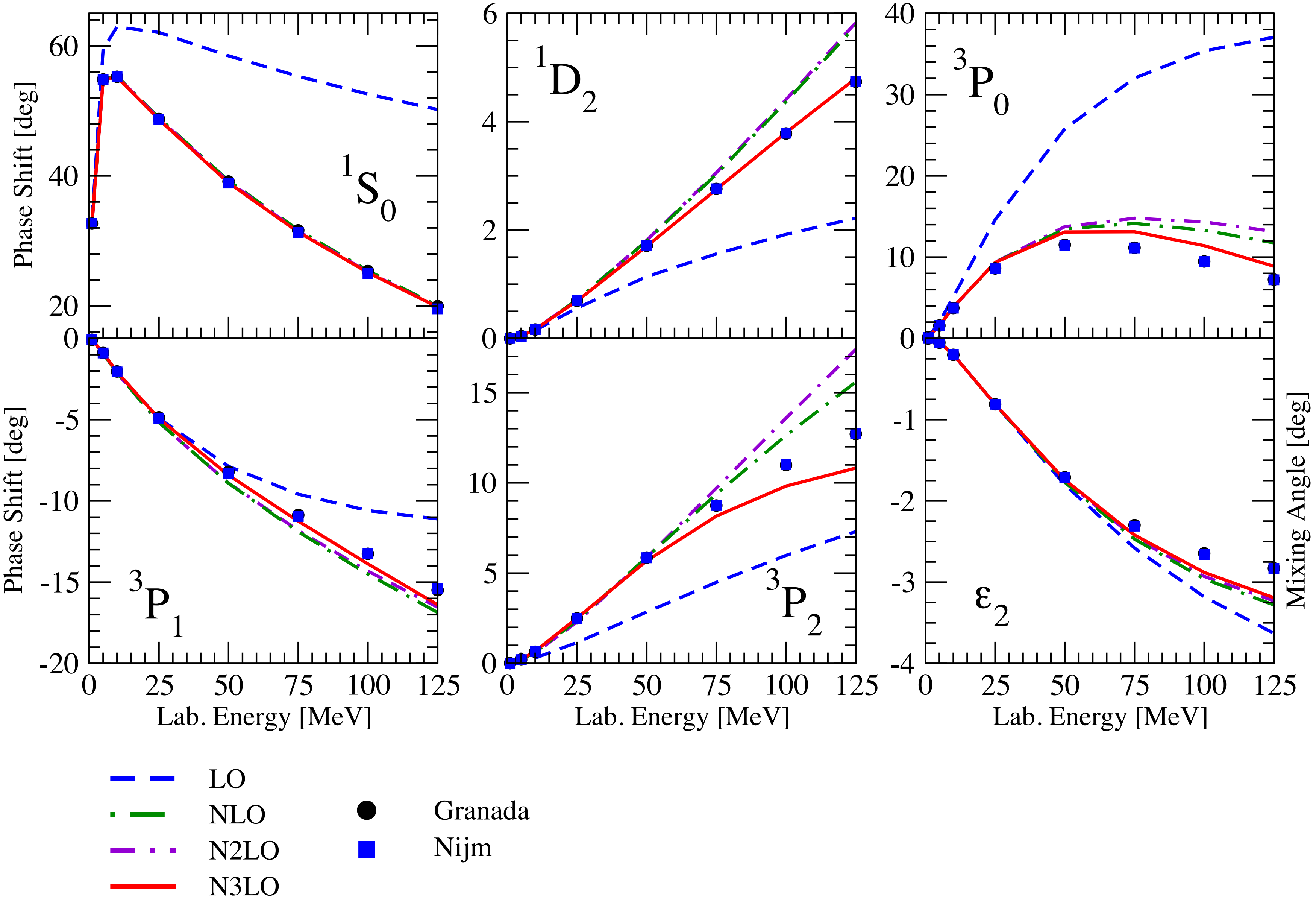
S-wave, P-wave, D-wave phase shifts in the np $T = 1$ channel (order by order model b up to 125 MeV)



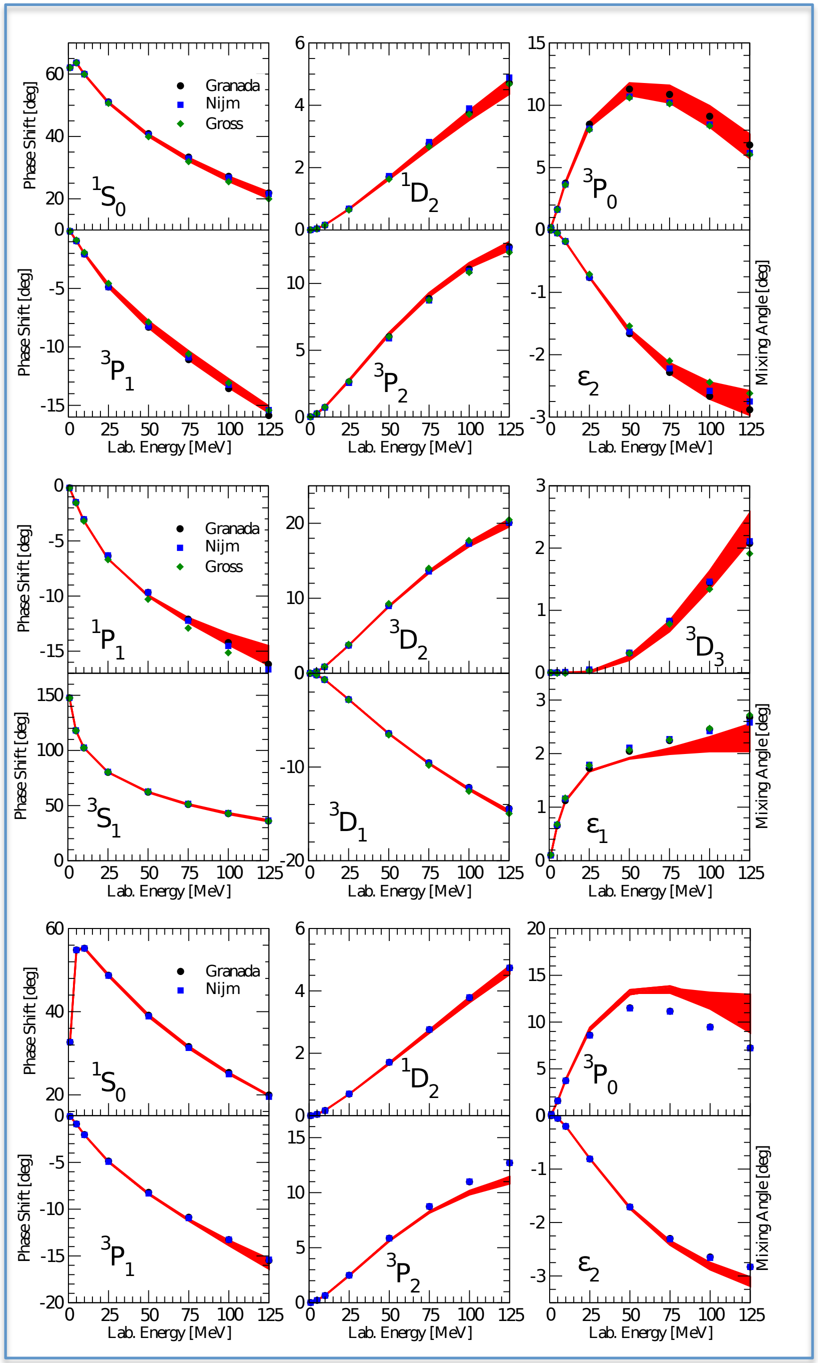
S-wave, P-wave, D-wave phase shifts in the np $T = 0$ channel (order by order model b up to 125 MeV)

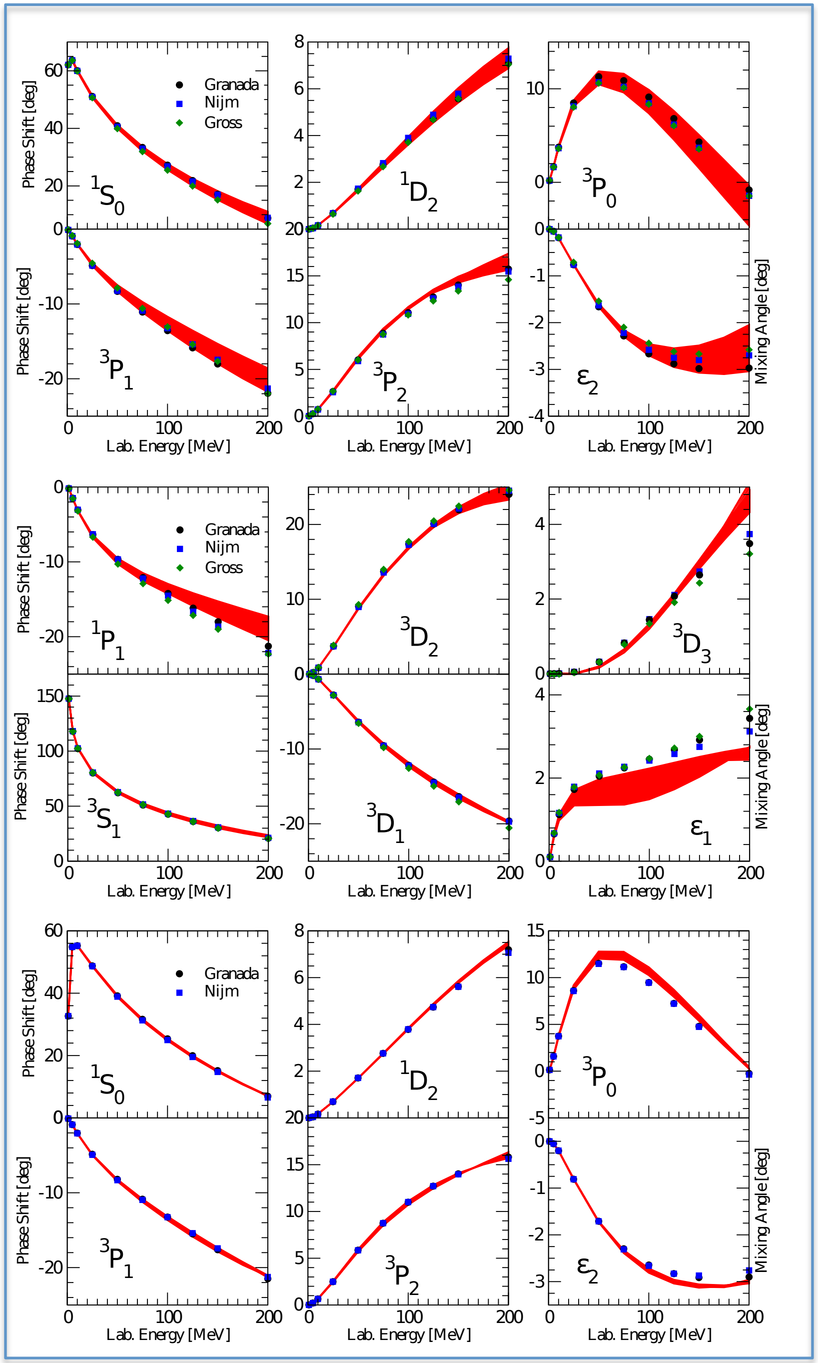


S-wave, P-wave, D-wave phase shifts in the *pp* $T = 1$ channel (order by order model b up to 125 MeV)



Phase Shifts: 125 and 200 MeV





Variational Monte Carlo: VMC

R. B. Wiringa Phys. Rev. C **43**, 1585 (1991)

- Minimize the expectation value of H

$$\boxed{E_V = \frac{\langle \Psi_V | H | \Psi_V \rangle}{\langle \Psi_V | \Psi_V \rangle} \geq E_0}$$

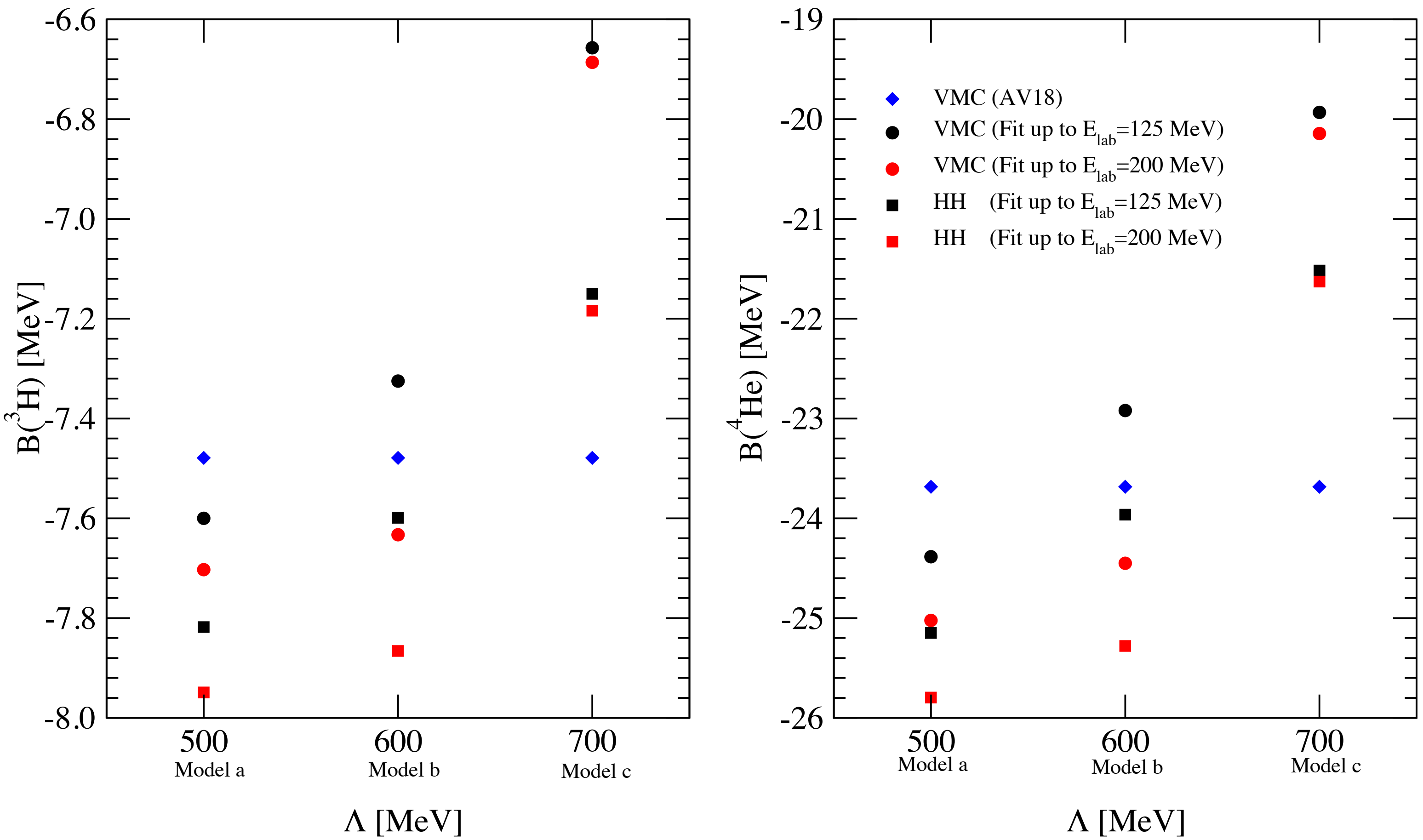
using Metropolis algorithm and trial function (for s-shell nuclei)

$$\boxed{|\Psi_V\rangle = \left[S \prod_{i<j} (1 + U_{ij}) \right] \underbrace{\left[\prod_{i<j<k} f^c_{ijk}(\mathbf{r}_{ik}\,, \mathbf{r}_{jk}) \prod_{i<j} f_c(\mathbf{r}_{ij}) \right] |\Phi(JMTT_z)\rangle}_{|\Psi_J\rangle \,=\, \text{Jastrow wave function}}}$$

- $|\Phi(JMTT_z)\rangle$ is an antisymmetric product of sigle-particle wave functions with the designed (J^π, T)
- $f_c(\mathbf{r}_{ij})$ and $f^c_{ijk}(\mathbf{r}_{ik}\,, \mathbf{r}_{jk})$ are central (spin-isospin independent) two- and three- body correlations
- $S \prod_{i<j}$ represents a symmetrized product
- pair correlation operators $U_{ij} = \sum_{p=2,6} \left[\prod_{k \neq i,j} f^p_{ijk}(\mathbf{r}_{ik}\,, \mathbf{r}_{ik}) \right] u_p(\mathbf{r}_{ij})\, O^p_{ij}$

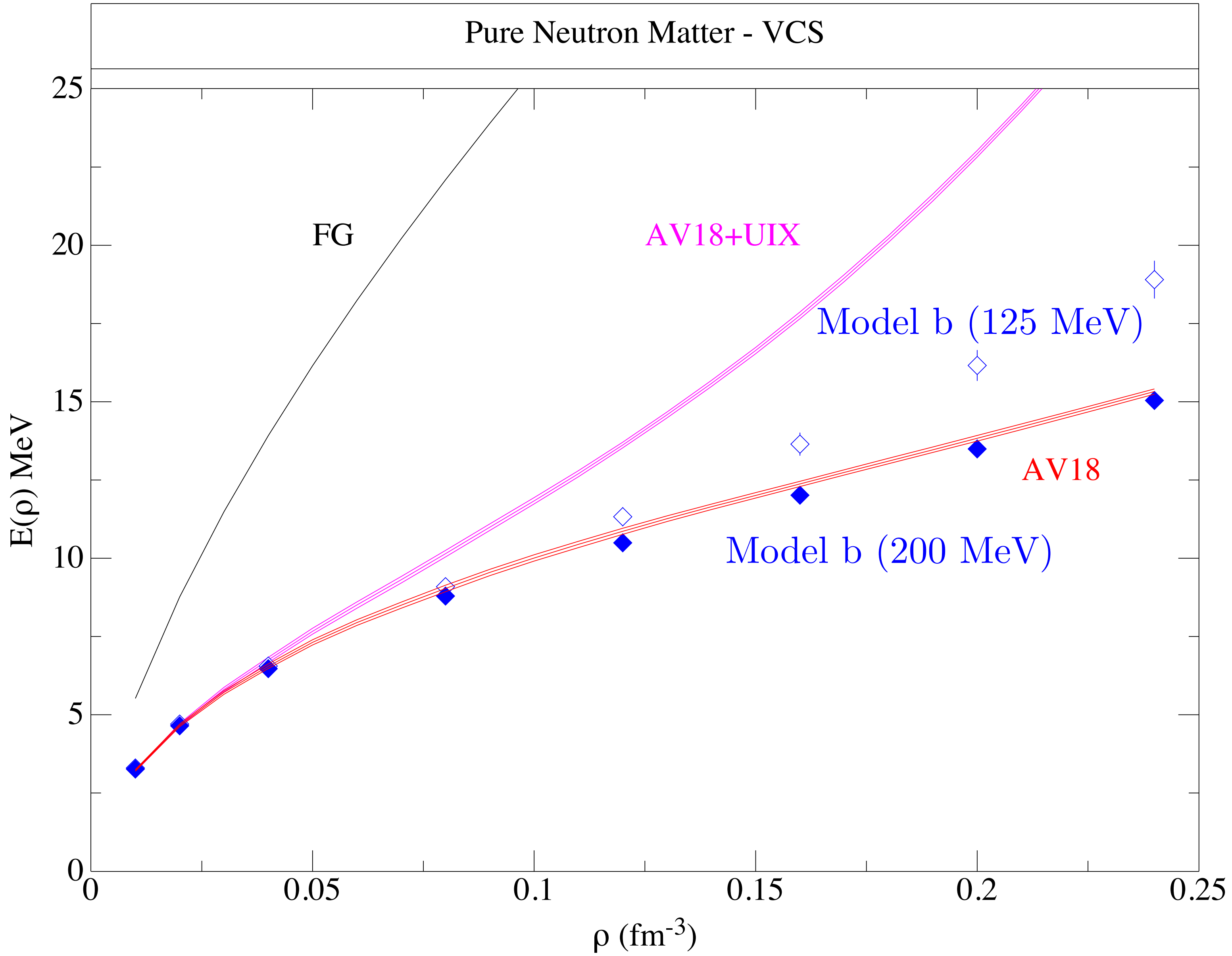
$$O^{p=1,6}_{ij} = [\mathbf{1},\, \boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_j,\, S_{ij}] \otimes [\mathbf{1},\, \boldsymbol{\tau}_i \cdot \boldsymbol{\tau}_j]$$

- ❖ To perform minimizations of ground-state energies with these chiral potentials a minimizer has been implemented recently in the VCM code
  - ➢ NLopt is a free/open-source library for non-linear optimization (http://ab-initio.mit.edu/wiki/index.php/NLopt)



- ❖ HH calculations provided by the Pisa group (Marcucci, Kievsky, Viviani)

Preliminary: Pure Neutron Matter



Courtesy of
R. Wiringa

Summary:

- We construct a family of local NN potential with chiral TPE including Δ-isobar up to N2LO (Q^3) and contact interactions up to N3LO(Q^4)
- Three versions of this chiral potential for three different cutoffs have been developed with fits to np and pp data up to $E_{\mathrm{lab}} = 125$ MeV and 200 MeV, deuteron binding energy and nn scattering length
- We provide some preliminary results using these potentials in light-structure nuclei and pure neutron matter

Plans:

- Implementation of these chiral potentials in the GFMC
- Development of a consistent three-body force (in collaboration with A. Baroni–ODU and S. Pastore–LANL)

Nuclear χEFT Approach:

S. Weinberg, Phys. Lett. **B251**, 288 (1990); Nucl. Phys. **B363**, 3 (1991); Phys. Lett. **B295**, 114 (1992)

- χEFT uses the chiral-symmetry to constrain the interactions of π's among themselves or with baryons (N and Δ-isobar)

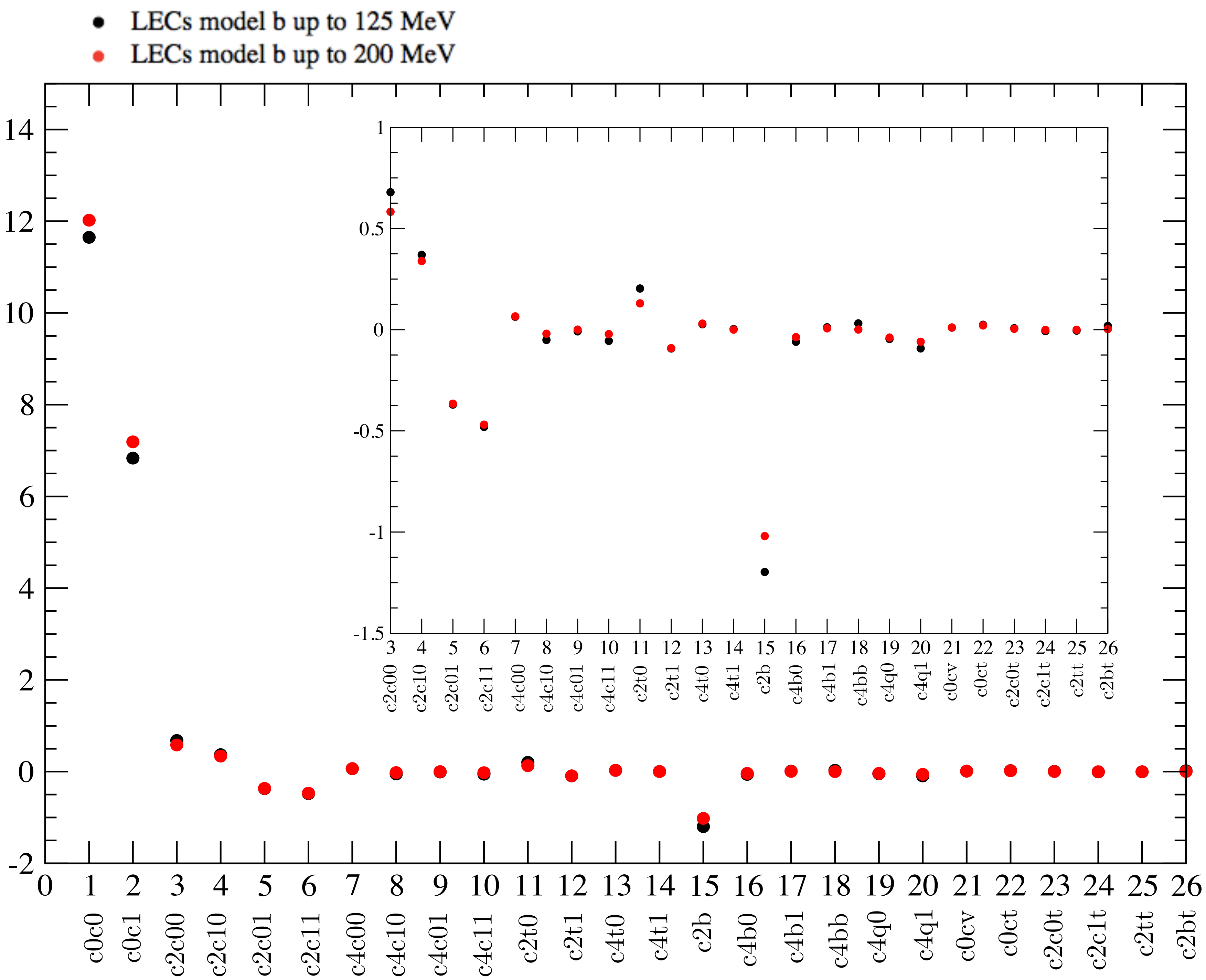
- π's couple by powers of its momentum Q, and the Lagrangian ($\mathcal{L}_{\text{eff}}$) can be expanded systematically in powers of Q/Λ; ($Q \ll \Lambda \sim 1\,\text{GeV}$ is the chiral-symmetry breaking scale and $Q \sim m_\pi$)

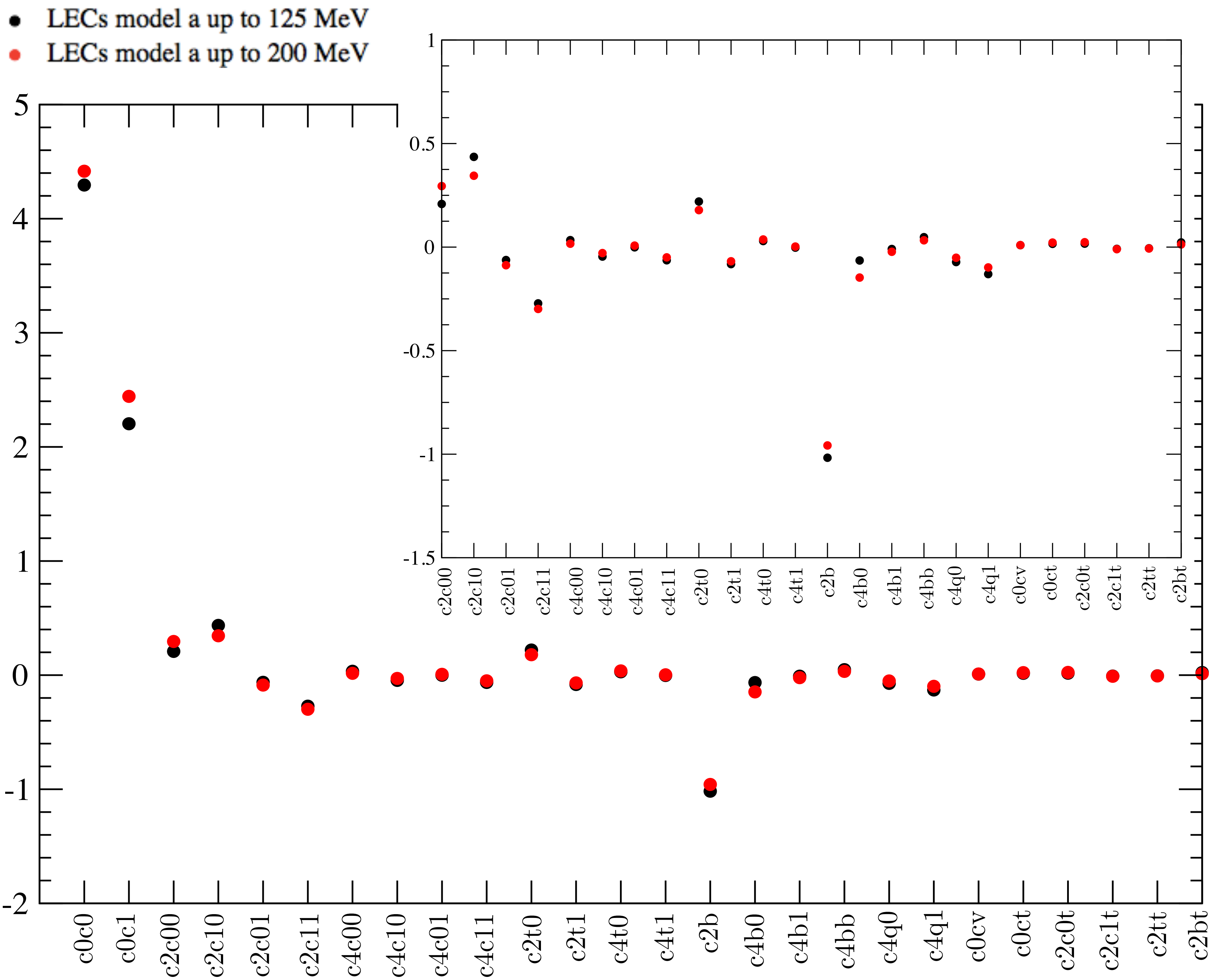
$$\boxed{\mathcal{L}_{eff} = \mathcal{L}^{(0)} + \mathcal{L}^{(1)} + \mathcal{L}^{(2)} + \ldots}$$

- χEFT allows for a perturbative treatment in terms of powers of Q

- The unknown coefficients of the perturbative expansion are called LEC's and are determined fitting the experimental data

- The χ-expansion gives rise to potentials and external currents can be naturally incorporated


● LECs model b up to 125 MeV
● LECs model b up to 200 MeV
14
12
10
8
6
4
2
0
-2
0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26
c0c0 c0c1 c2c00 c2c10 c2c01 c2c11 c4c00 c4c10 c4c01 c4c11 c2t0 c2t1 c4t0 c4t1 c2b c4b0 c4b1 c4bb c4q0 c4q1 c0cv c0ct c2c0t c2c1t c2tt c2bt
1
0.5
0
-0.5
-1
-1.5
3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26
c2c00 c2c10 c2c01 c2c11 c4c00 c4c10 c4c01 c4c11 c2t0 c2t1 c4t0 c4t1 c2b c4b0 c4b1 c4bb c4q0 c4q1 c0cv c0ct c2c0t c2c1t c2tt c2bt



● LECs model a up to 125 MeV
● LECs model a up to 200 MeV
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c2c01
c2c11
c4c00
c4c10
c4c01
c4c11
c2t0
c2t1
c4t0
c4t1
c2b
c4b0
c4b1
c4bb
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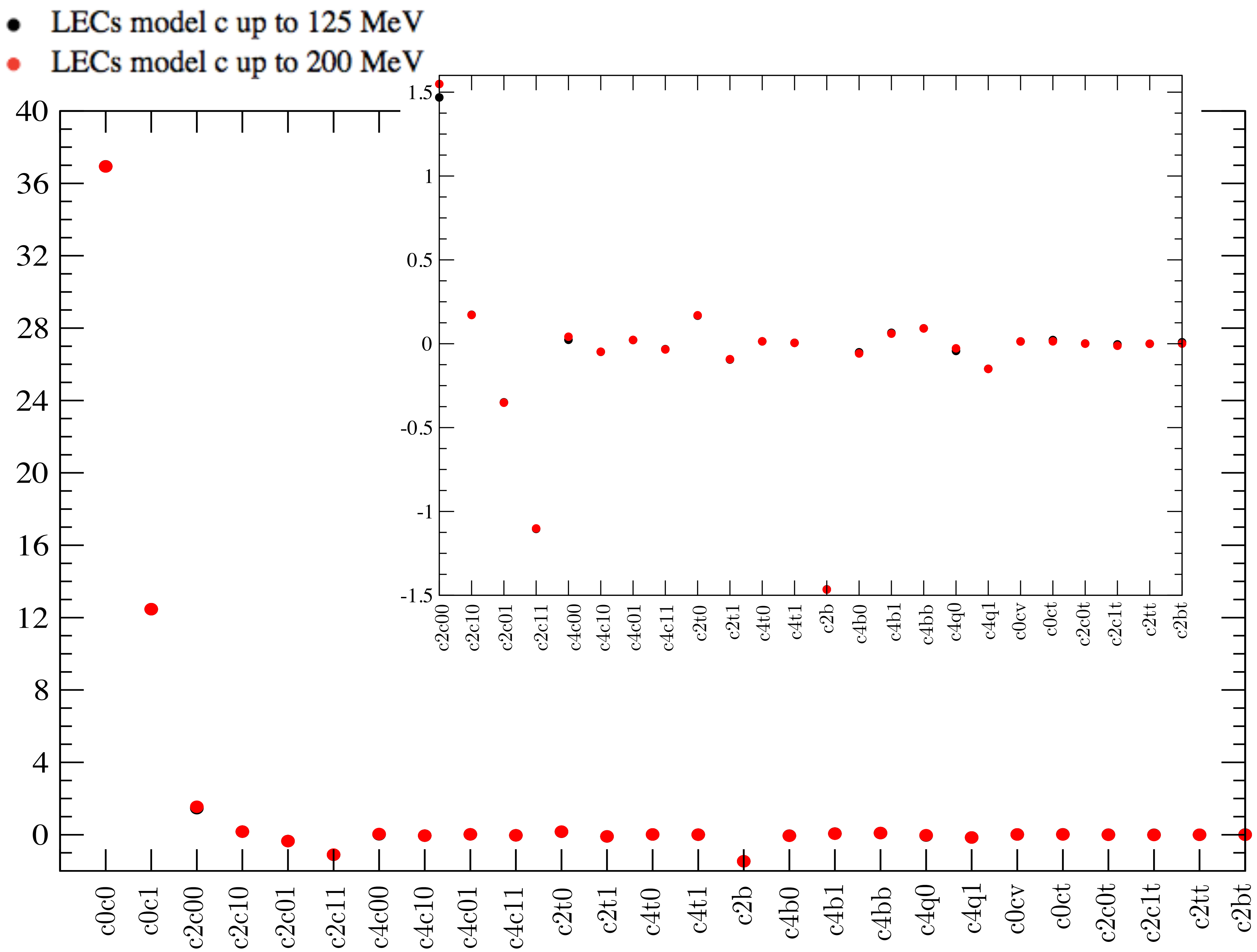
● LECs model c up to 125 MeV
● LECs model c up to 200 MeV
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c2c10
c2c01
c2c11
c4c00
c4c10
c4c01
c4c11
c2t0
c2t1
c4t0
c4t1
c2b
c4b0
c4b1
c4bb
c4q0
c4q1
c0cv
c0ct
c2c0t
c2c1t
c2tt
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-0.5
-1
-1.5
c2c00
c2c10
c2c01
c2c11
c4c00
c4c10
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c4b0
c4b1
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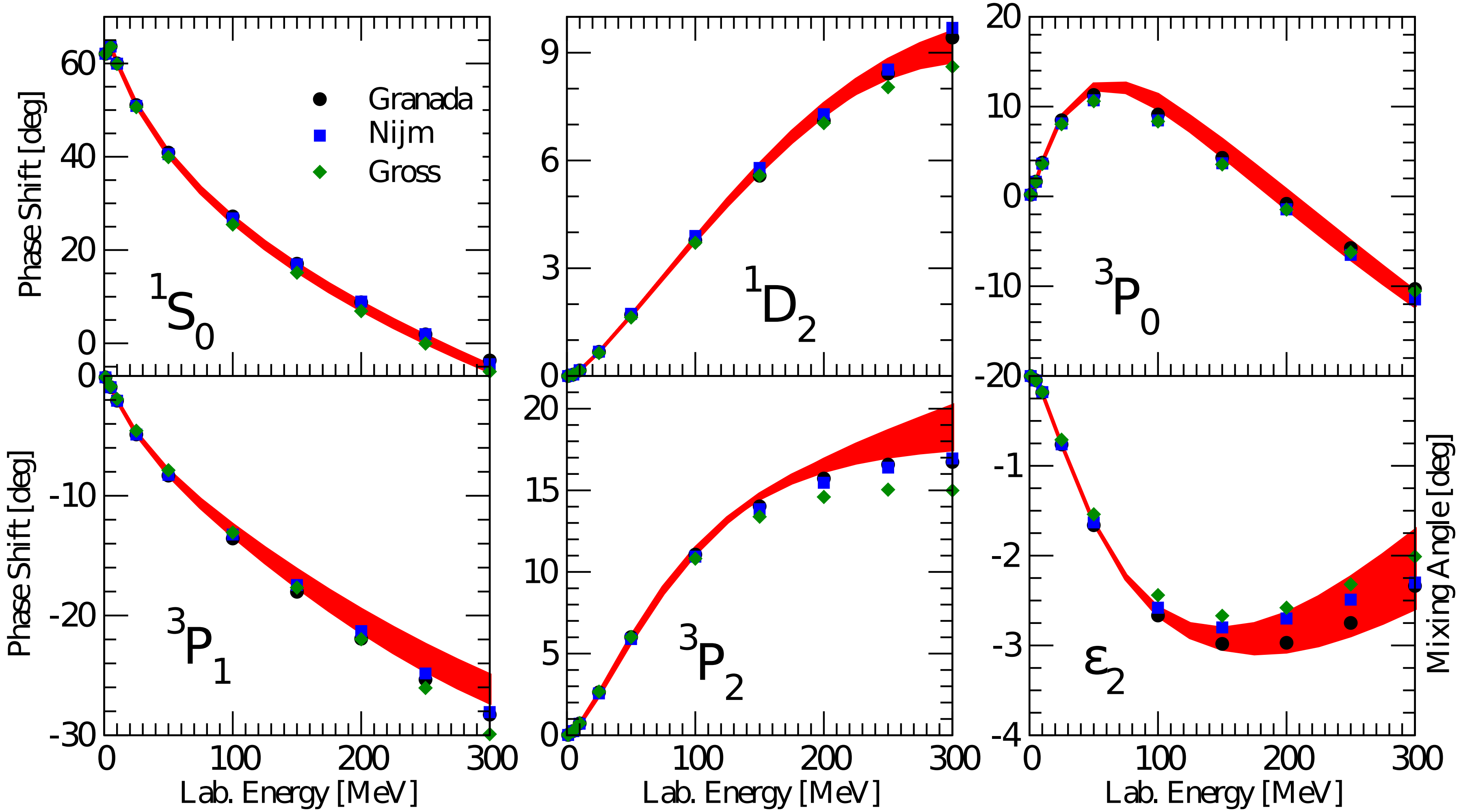
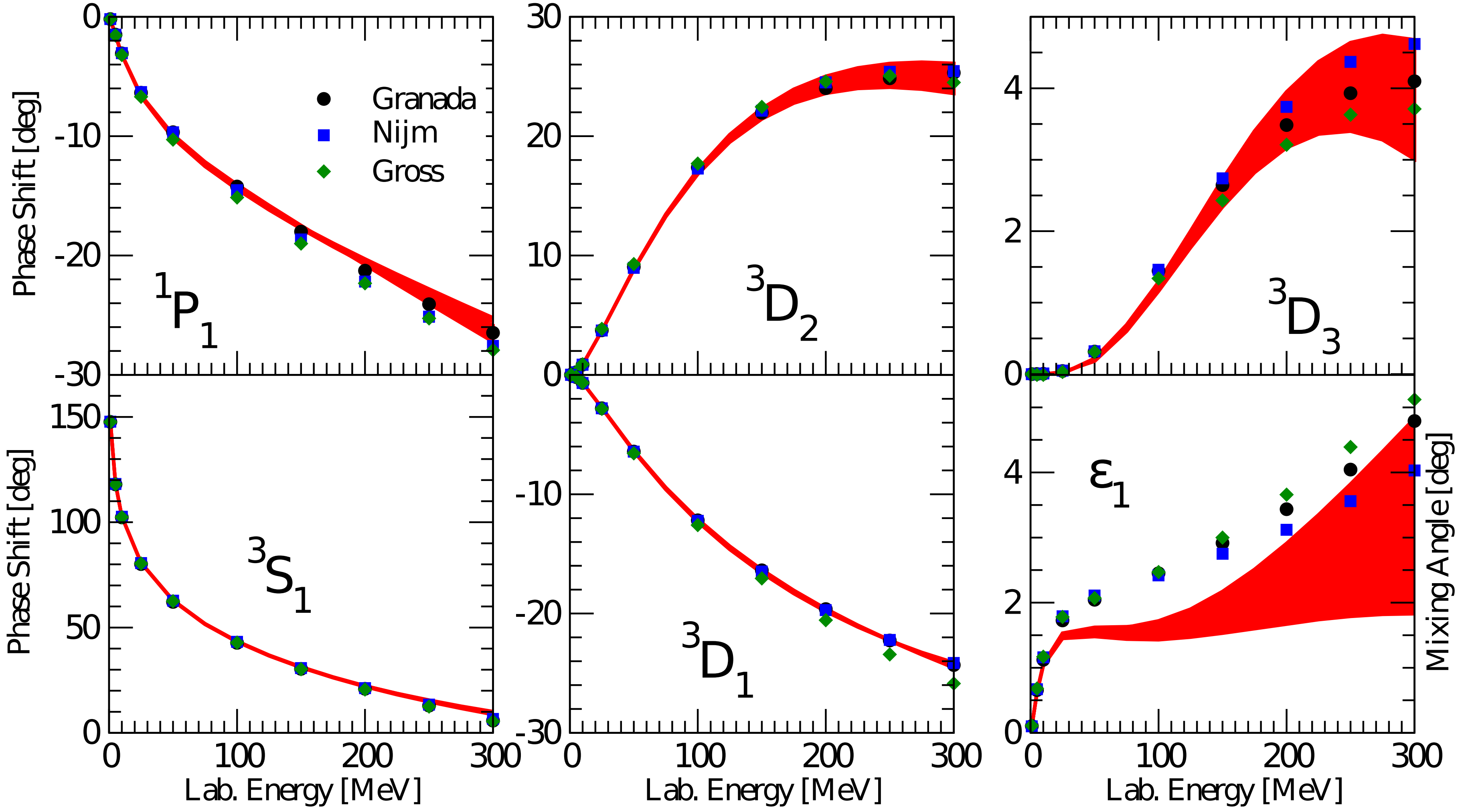
Extra Slides

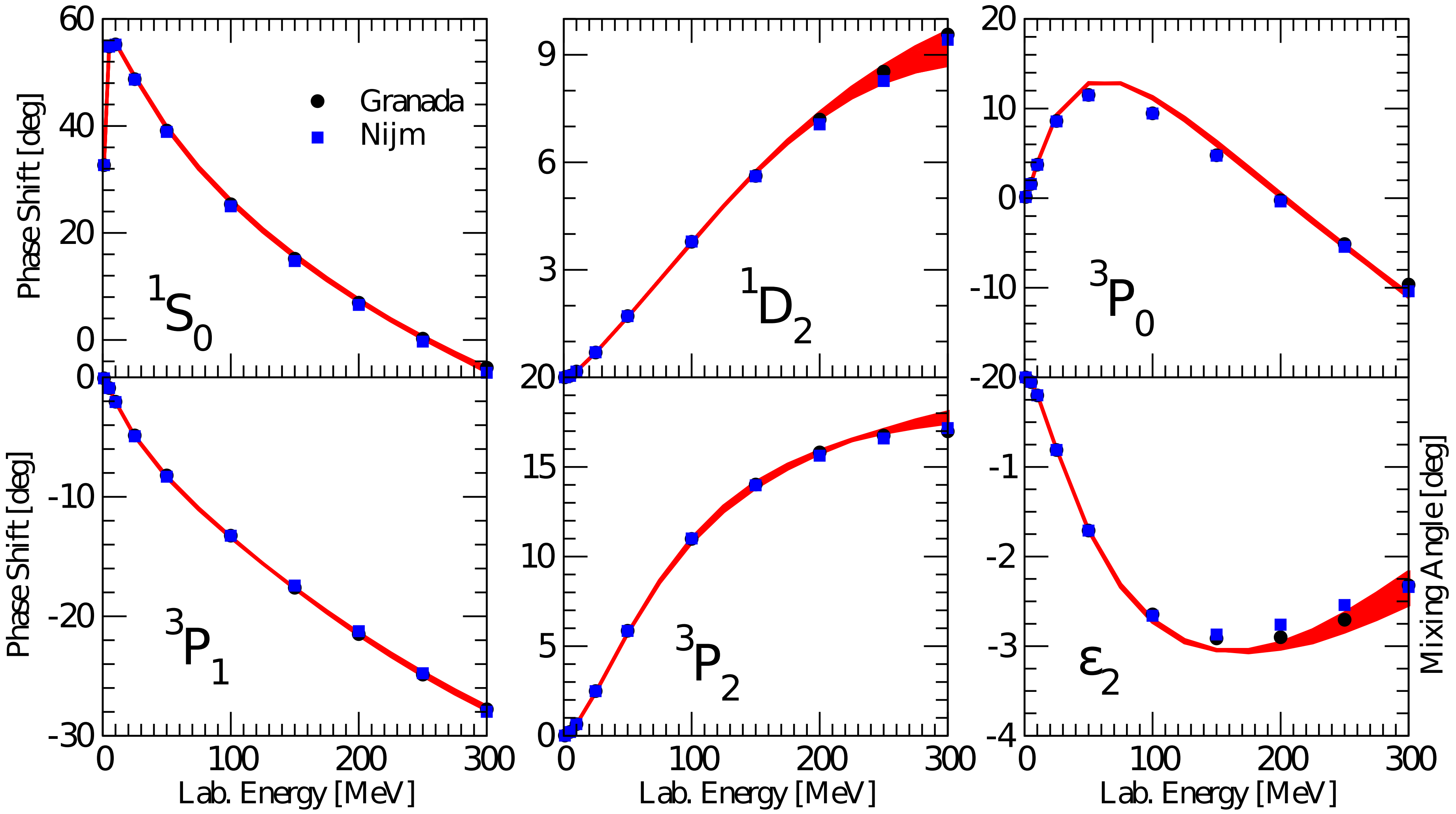
S-wave, P-wave, and D-wave phase shifts in the np T=1 channel



S-wave, P-wave, and D-wave phase shifts in the np $T=0$ channel



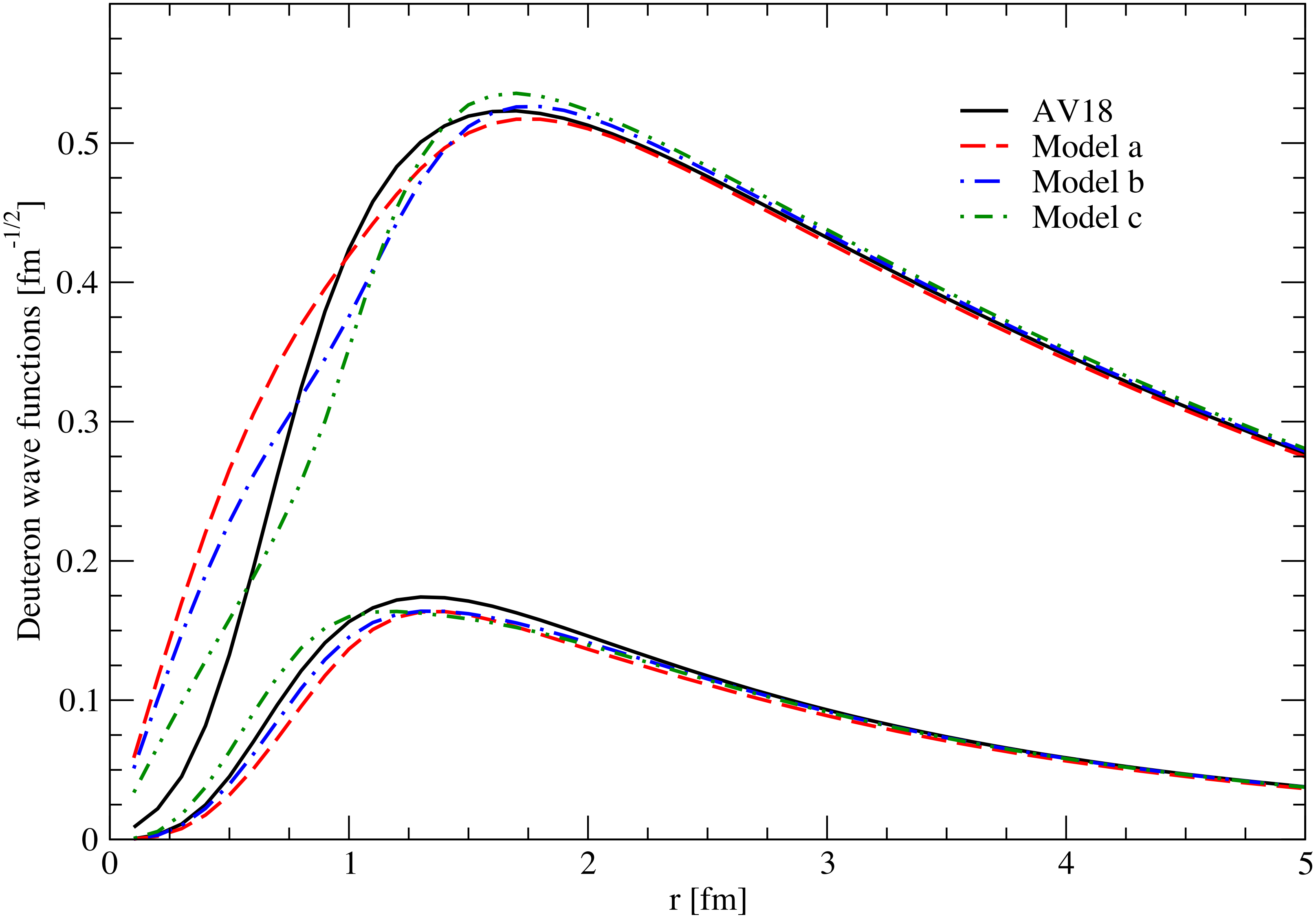
S-wave, P-wave, and D-wave phase shifts in the *pp* T=1 channel



Outline

- Nuclear χEFT approach
- General considerations
- Construction of a family of minimally nonlocal/local chiral potentials
- Fitting procedure:
  - 2013 Granada NN database
  - np and pp phase shifts analysis
- VMC and HH for ^{3}H and ^{4}He
- Summary

The S-wave and D-wave components of the deuteron wave function corresponding to models a (dashed lines), b (dotted-dashed lines) and c (dotted-dashed-dotted lines) are compared with those corresponding to the AV18 (solid lines)



Deuteron static properties for the three models of chiral potential with $(R_{\rm L}, R_{\rm S})$=(1.2,0.8) fm (model a), (1.0,0.7) fm (model b), and (0.8,0.6) fm (model c)

	Experiment	$v_{12}^{\rm a}$	$v_{12}^{\rm b}$	$v_{12}^{\rm c}$
E_d [MeV]	2.224575(9)	2.224575	2.224574	2.224575
η	0.0256(4)	0.0245	0.0248	0.0246
r_d [fm]	1.97535(85)	1.948	1.975	1.989
μ_d $[\mu_0]$	0.857406(1)	0.852	0.850	0.848
Q_d [fm^2]	0.2859(3)	0.257	0.268	0.269
P_d [%]		4.94	5.29	5.55

The singlet and triplet *np*, and singlet *pp*, scattering lengths and effective ranges for the three models of chiral potential with ($R_{\rm L}, R_{\rm S}$)=(1.2,0.8) fm (model a), (1.0,0.7) fm (model b), and (0.8,0.6) fm (model c)

	Experiment	v_{12}^a	w/o $v_{12}^{\rm EM}$	v_{12}^b	w/o $v_{12}^{\rm EM}$	v_{12}^c	w/o $v_{12}^{\rm EM}$
$^1a_{pp}$	$-7.8063(26)$ $-7.8016(29)$	-7.766	-17.014	-7.766	-16.956	-7.763	-17.137
$^1r_{pp}$	2.794(14) 2.773(14)	2.742	2.818	2.743	2.820	2.730	2.802
$^1a_{nn}$	$-18.90(40)$	-18.867	-19.148	-19.025	-19.301	-18.719	-19.039
$^1r_{nn}$	2.75(11)	2.831	2.827	2.799	2.795	2.738	2.732
$^1a_{np}$	$-23.740(20)$	-23.752	-23.196	-23.755	-23.248	-23.745	-23.167
$^1r_{np}$	2.77(5)	2.665	2.670	2.672	2.677	2.638	2.644
$^3a_{np}$	5.419(7)	5.408	5.391	5.404	5.389	5.412	5.396
$^3r_{np}$	1.753(8)	1.741	1.740	1.737	1.734	1.740	1.745

Motivations

❖ Quantitative knowledge of NN forces is crucial in order to understand the properties of nuclei and nuclear matter

❖ QCD:
- quarks, gluons
- weak at short distance (asymptotic freedom)
- strong at long distance ($\geq$ 1 fm) or low energies (low-energy QCD)

↓

❖ Nuclear forces: difficult to derive in terms of quarks and gluons

❖ EFT applied to low-energy QCD (χEFT):
- pions, nucleons, Δ's,..
- construct their interactions consistently with the symmetries of the underlying theory

Formalism: Transition Amplitude in TOPT

I. Nucleon-nucleon potential ($NN \rightarrow NN$)

- Degrees of freedom: non-relativistic N's and Δ's, relativistic π's

- Time-ordered perturbation theory (TOPT)

$$\langle N'N'|T|NN\rangle = \langle N'N'|H_1 \sum_{n=1}^{\infty} \left(\frac{1}{E_i - H_0 + i\eta} H_1 \right)^{n-1} |NN\rangle$$

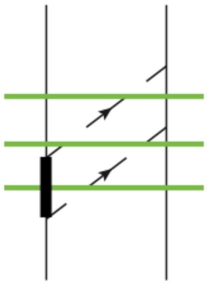
H_1 = interaction Hamiltonians among π, *N*, and Δ implied by $\mathcal{L}_{eff}$

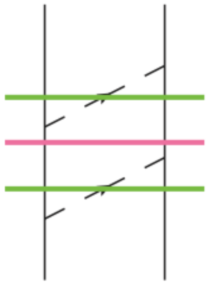
H_0 = free π, *N*, and Δ Hamiltonians

- Completeness: $\sum_{I_i} |I_i\rangle\langle I_i| = 1$ between successive terms of H_1

$$\begin{aligned}\langle f|T|i\rangle &= \langle f|H_1|i\rangle + \sum_{I_1} \langle f|H_1|I_1\rangle \frac{1}{E_i - E_1 + i\eta} \langle I_1|H_1|i\rangle \\ &+ \sum_{I_1,I_2} \langle f|H_1|I_2\rangle \frac{1}{E_i - E_2 + i\eta} \langle I_2|H_1|I_1\rangle \frac{1}{E_i - E_1 + i\eta} \langle I_1|H_1|i\rangle + \ldots.\end{aligned}$$

Formalism con't


Irreducible


Reducible

- Two kinds of diagrams: reducible and irreducible
- N vertices represented by $\langle I_j | H_1 | I_k \rangle$
- N-1 energy denominators $(E_i - E_k + i\eta)^{-1}, k = 1, ..., (N-1)$
  - N_K denominators involving only nucleonic energies scales as Q^{-2}
  - N-N_K-1 denominators involving nucleons, π's, and Δ's energies
- A contribution with N interaction vertices and L loops scales as

$$m = \underbrace{\prod_{i=1}^{N} Q^{\alpha_i - \beta_i/2}}_{H_1 \text{ scaling}} \times \underbrace{Q^{-(N-N_K-1)} Q^{-2N_K}}_{\text{denominators}} \times \underbrace{Q^{3L}}_{\text{Loop integration}}$$

- A N-N$_K$-1 denominators can be further expanded:

$$\frac{1}{E_i - E_j + i\eta} \equiv \frac{1}{E_i - E_{I_j} - \Omega + i\eta} = -\frac{1}{\Omega}\left[1 + \frac{E_i - E_{I_j}}{\Omega} + \frac{(E_i - E_{I_j})^2}{\Omega^2} + \ldots\right]$$

  - E_{I_j} : kinetic energies of intermediate states (2N’s or 1N+1Δ or 2Δ)
  - $\Omega \equiv \omega_\pi$ (if one or more pion are involved)

    $\Omega \equiv \omega_\pi + \Delta$ (if one or more pion are involved and a Δ), etc…

$$\Delta = m_\Delta - m_N \sim 300\,\text{MeV} \sim 2m_\pi$$

- The expansion in power of Q is: $-\dfrac{1}{Q}\Big[\underset{\substack{\text{static limit}\\ m_N,\, m_\Delta \to \infty}}{\textcircled{1}} + \underbrace{Q + Q^2 + \ldots..}_{\text{non-static corrections}}\Big]$

- In chiral-expansion T -matrix can be expanded as:

$$T = T^{(0)} + T^{(1)} + T^{(2)} + \ldots \qquad \text{with} \qquad T^{(m)} \sim Q^{(m)}$$

From Amplitudes to Potentials

❖ Construct potential v such that when inserted in Lippmann-Schwinger (LS) equation

$$v + vG_0v + vG_0vG_0v + \ldots.$$

$G_0 =$ two-nucleon propagator (Q^{-2})

$$G_0 = 1/(E_i + E_I + i\eta)$$

leads to T-matrix order by order in the power counting

- Assume: $v = v^{(0)} + v^{(1)} + v^{(2)} \ldots$ (with $v^{(m)} \sim Q^m$)

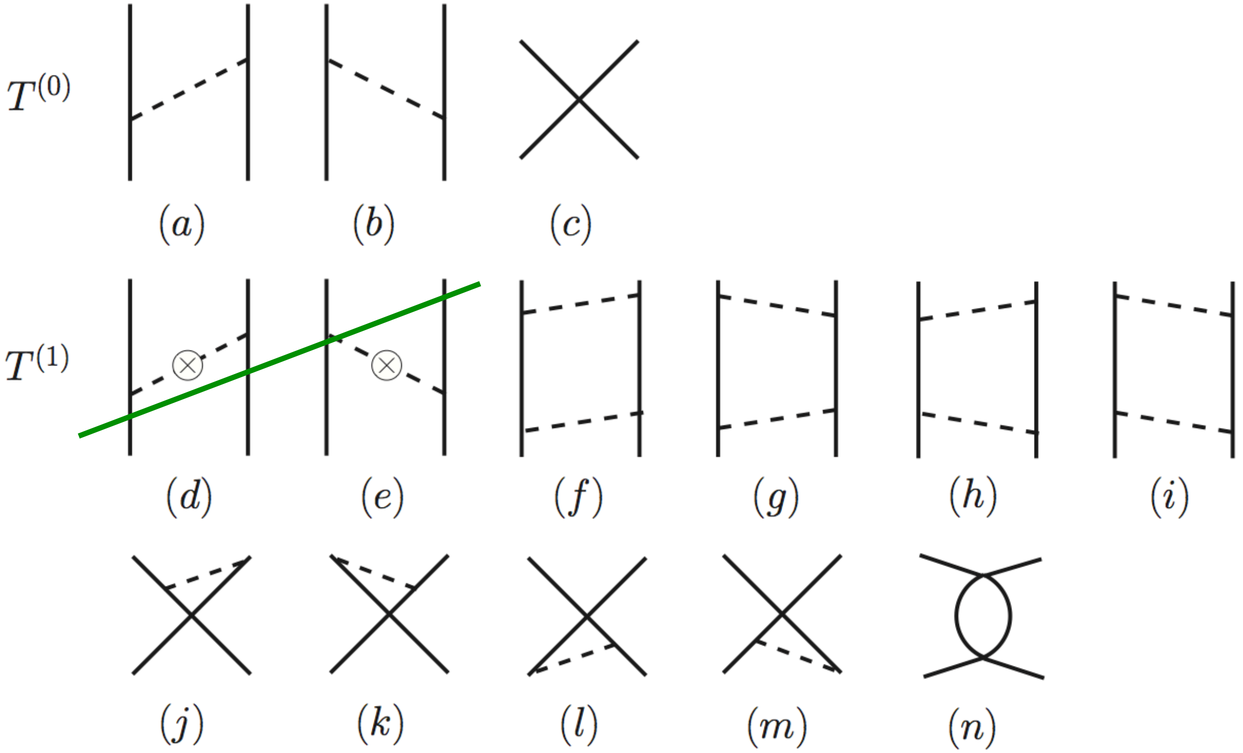
- Matching expansion for T with the LS equation order by order:

$$\begin{aligned}
v^{(0)} &= T^{(0)} \\
v^{(1)} &= T^{(1)} - \left[v^{(0)}G_0v^{(0)}\right] \\
v^{(2)} &= T^{(2)} - \left[v^{(0)}G_0v^{(0)}G_0v^{(0)}\right] - \left[v^{(1)}G_0v^{(0)} + v^{(0)}G_0v^{(1)}\right] \\
v^{(3)} &= T^{(3)} - \left[v^{(0)}G_0v^{(0)}G_0v^{(0)}G_0v^{(0)}\right] - \left[v^{(1)}G_0v^{(0)}G_0v^{(0)} + \text{permutations}\right] \\
&\quad - \left[v^{(1)}G_0v^{(1)}\right] - \left[v^{(2)}G_0v^{(1)} + v^{(0)}G_0v^{(2)}\right]
\end{aligned}$$

- A term like $v^{(m)}G_0v^{(n)} \sim Q^{m+n+1}$

$v^{(m)}$ up to order m=1

- Time-ordered diagrams contributing to the χEFT T-matrix up to order Q^1



- $v^{(0)} \quad = \quad T^{(0)}$ consists of (static) OPE and contact terms ⇒ (LO)
- $v^{(1)} \quad = \quad T^{(1)} - \left[v^{(0)} G_0 v^{(0)} \right]$ vanishes

II. Charge/Current operators ($NN\gamma \rightarrow NN$)

- Similar prescription for potential $v_\gamma = A^\mu J_\mu = A^0 \rho - \mathbf{A} \cdot \mathbf{J}$

- Power counting of the EM interaction (treated in first order)

$$T_\gamma = T_\gamma^{(-3)} + T_\gamma^{(-2)} + T_\gamma^{(-1)} \dots$$

- In the context of LS: $v_\gamma = v_\gamma^{(-3)} + v_\gamma^{(-2)} + v_\gamma^{(-1)} \dots$
  - Matching expansion for T with the LS equation order by order:

$$\begin{aligned}
v_\gamma^{(-3)} &= T_\gamma^{(-3)} \, , \\
v_\gamma^{(-2)} &= T_\gamma^{(-2)} - \left[v_\gamma^{(-3)} \, G_0 \, v^{(0)} + v^{(0)} \, G_0 \, v_\gamma^{(-3)} \right] \, , \\
v_\gamma^{(-1)} &= T_\gamma^{(-1)} - \left[v_\gamma^{(-3)} \, G_0 \, v^{(0)} \, G_0 \, v^{(0)} + \text{permutations} \right] \\
&\quad \underbrace{- \left[v_\gamma^{(-2)} \, G_0 \, v^{(0)} + v^{(0)} \, G_0 \, v_\gamma^{(-2)} \right]}_{\text{LS terms}}
\end{aligned}$$

$$\cdots\cdots$$

- Charge and current operators up to one loop (eQ) consistent with v at Q^2

Technical Issue

- ❖ Ultraviolet divergencies associated with nuclear and electromagnetic loop diagrams are removed via dimensional regularization (DR) and the divergent part of these loops are absorbed in the redefinition of the relevant LEC's

- ❖ Resulting renormalized operators have power-law behavior for large momenta:

  further regularization needs to be employed before these operators can be for solving Schrödinger equation and for calculation of current matrix elements → cutoff functions C_Λ

The Basic Model

- The nucleus is a system made of A interacting particles described by

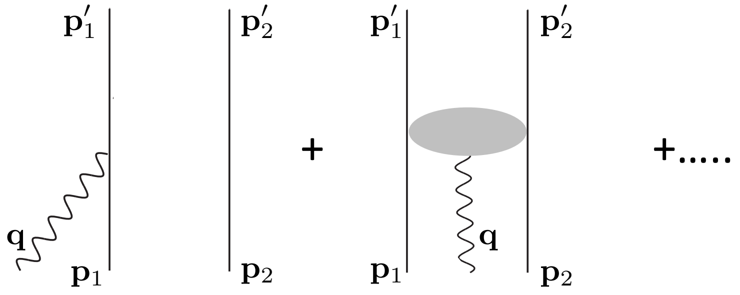
$$H = T + V = \sum_{i=1}^{A} t_i + \sum_{i<j} v_{ij} + \sum_{i<j<k} V_{ijk} + ...$$

where v_{ij} and V_{ijk} are the 2- 3-nucleon interaction operators

- Current and charge operators describe the interaction of nuclei with external fields. They are expanded as sum of 2-, 3-,…nucleon operators:

$$\rho = \sum_{i=1}^{A} \rho_i + \sum_{i<j} \rho_{ij} + ...$$

$$\mathbf{j} = \sum_{i=1}^{A} \mathbf{j}_i + \sum_{i<j} \mathbf{j}_{ij} + ...$$



- Longitudinal EM current operator **j** linked to the nuclear Hamiltonian via

$$\mathbf{q} \cdot \mathbf{j} = [H,\, \rho] = [T + V,\, \rho]$$

The $v^{\mathrm{EM}}(pp)$

$$v^{EM}(pp) = V_{C1}(pp) + V_{C2}(pp) + V_{DF}(pp) + V_{VP}(pp) + V_{MM}(pp)$$

Leading Coulomb interaction $V_{C1}(pp) = \alpha' \frac{F_C(r)}{r}$,

Second order Coulomb interaction $V_{C2}(pp) = -\frac{\alpha}{2M_p^2}\left[(\nabla^2 + k^2)\frac{F_C(r)}{r} + \frac{F_C(r)}{r}(\nabla^2 + k^2)\right] \approx -\frac{\alpha\alpha'}{M_p}\left[\frac{F_C(r)}{r}\right]^2$,

Darwin-Foldy interaction $V_{DF}(pp) = -\frac{\alpha}{4M_p^2}F_\delta(r)$,

Vacuum polarization interaction $V_{VP}(pp) = \frac{2\alpha\alpha'}{3\pi}\frac{F_C(r)}{r}\int_1^\infty dx\, e^{-2m_e r x}\left[1 + \frac{1}{2x^2}\right]\frac{(x^2-1)^{1/2}}{x^2}$,

Magnetic moment interaction $V_{MM}(pp) = -\frac{\alpha}{4M_p^2}\mu_p^2\left[\frac{2}{3}F_\delta(r)\boldsymbol{\sigma}_i\cdot\boldsymbol{\sigma}_j + \frac{F_t(r)}{r^3}S_{ij}\right] - \frac{\alpha}{2M_p^2}(4\mu_p - 1)\frac{F_{ls}(r)}{r^3}\mathbf{L}\cdot\mathbf{S}$

- $\alpha' \equiv 2k\alpha/M_p v_{lab}$ takes into account the energy dependence of the Coulomb interaction via relativistic effects): k is the relative momentum in the COM frame, M_p proton mass and v_{lab} is the velocity in the LAB frame (significantly different from α at even moderate energies ~20% difference at T_{lab}=250 MeV), , μ_n and μ_p neutron and proton magnetic moments
- $F_C(r)=$ FT of the Sacks form factors $G_E^p = \frac{G_M^p}{\mu_p} = \frac{G_M^n}{\mu_n} = G_D = \left(1 + \frac{q^2}{b^2}\right)^{-2}$; $b = 4.27\,\mathrm{fm}^{-1}$

  $x = br$

$$F_C(r) = 1 - \left(1 + \frac{11}{16}x + \frac{3}{16}x^2 + \frac{1}{48}x^3\right)e^{-x},$$

$$F_\delta(r) = b^3\left(\frac{1}{16} + \frac{1}{16}x + \frac{1}{48}x^2\right)e^{-x},$$

$$F_t(r) = 1 - \left(1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{144}x^5\right)e^{-x},$$

$$F_{ls}(r) = 1 - \left(1 + x + \frac{1}{2}x^2 + \frac{7}{48}x^3 + \frac{1}{48}x^4\right)e^{-x}$$

The $v^{\mathrm{EM}}(np)$

$$v^{EM}(np) = V_{C1}(np) + V_{MM}(np)$$

Leading Coulomb interaction $V_{C1}(np) = \alpha\beta_n \frac{F_{np}(r)}{r}$

Magnetic moment interaction
$$V_{MM}(np) = -\frac{\alpha}{4M_nM_p}\mu_n\mu_p\left[\frac{2}{3}F_\delta(r)\boldsymbol{\sigma}_i\cdot\boldsymbol{\sigma}_j + \frac{F_t(r)}{r^3}S_{ij}\right] - \frac{\alpha}{2M_nM_r}\mu_n\frac{F_{ls}(r)}{r^3}(\mathbf{L}\cdot\mathbf{S}+\mathbf{L}\cdot\mathbf{A})$$

- $F_{np}(r) =$ FT of the form factor $G_E^n = \beta_n q^2\left(1+\frac{q^2}{b^2}\right)^{-3}$, $\beta_n \equiv [dG_E^n/dq^2]_{q=0} = 0.0189\,\mathrm{fm}^2$
  $b = 4.27\,\mathrm{fm}^{-1}$
  $x = br$
  $$F_{np}(r) = b^2\left(15x + 15x^2 + 6x^3 + x^4\right)\frac{e^{-x}}{384}$$

- $\mathbf{A} = \frac{1}{2}(\boldsymbol{\sigma}_i - \boldsymbol{\sigma}_j)$ is the charge asymmetric operator which mixes spin-singlet and spin-triplet, M_r is the nucleon reduced mass, μ_n and μ_p neutron and proton magnetic moments, M_p and M_n proton and neutron masses.

The $v^{\mathrm{EM}}(nn)$

$$v^{EM}(nn) = V_{MM}(nn)$$

Magnetic moment interaction
$$V_{MM}(nn) = -\frac{\alpha}{4M_n^2}\mu_n^2\left[\frac{2}{3}F_\delta(r)\boldsymbol{\sigma}_i\cdot\boldsymbol{\sigma}_j + \frac{F_t(r)}{r^3}S_{ij}\right]$$

Momentum-space $v_{12}^{\rm S}$:

$$v_{12}^{\rm S}(\mathbf{k},\mathbf{K}) = v_{12}^{\rm S,CI}(\mathbf{k},\mathbf{K}) + v_{12}^{\rm S,CD}(\mathbf{k},\mathbf{K})$$

- ➤ $$\begin{aligned} v_{12}^{\rm S,CI}(\mathbf{k},\mathbf{K}) \;=\;& \left(C_S + C_1\, k^2 + D_1\, k^4\right) + \left(C_2\, k^2 + D_2\, k^4\right) \boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2 + \left(C_T + C_3\, k^2 + D_3\, k^4\right) \boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2 \\ &+ \left(C_4\, k^2 + D_4\, k^4\right) \boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2\, \boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2 + \left(C_5 + D_5\, k^2\right) S_{12}(\mathbf{k}) + \left(C_6 + D_6\, k^2\right) S_{12}(\mathbf{k})\, \boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2 \\ &+ i\left(C_7 + D_7\, k^2\right) \mathbf{S}\cdot(\mathbf{K}\times\mathbf{k}) + i\, D_8\, k^2\, \mathbf{S}\cdot(\mathbf{K}\times\mathbf{k})\, \boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2 + D_9\left[\mathbf{S}\cdot(\mathbf{K}\times\mathbf{k})\right]^2 + D_{10}\,(\mathbf{K}\times\mathbf{k})^2 \\ &+ D_{11}\,(\mathbf{K}\times\mathbf{k})^2\, \boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2 + D_{12}\, k^2 K^2 + D_{13}\, k^2 K^2 \boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2 + D_{14}\, K^2\, S_{12}(\mathbf{k}) \\ &+ D_{15}\, K^2\, S_{12}(\mathbf{k})\, \boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2 \end{aligned}$$
- ➤ $$\begin{aligned} v_{12}^{\rm S,CD}(\mathbf{k},\mathbf{K}) \;=\;& \left[C_0^{\rm IT} + C_1^{\rm IT}\, k^2 + C_2^{\rm IT}\, k^2\, \boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2 + C_3^{\rm IT}\, S_{12}(\mathbf{k}) + i\, C_4^{\rm IT} \mathbf{S}\cdot(\mathbf{K}\times\mathbf{k})\right] T_{12} \\ &+ \left[C_0^{\rm IV} + C_1^{\rm IV}\, k^2 + C_2^{\rm IV}\, k^2\, \boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2 + C_3^{\rm IV}\, S_{12}(\mathbf{k}) + i\, C_4^{\rm IV} \mathbf{S}\cdot(\mathbf{K}\times\mathbf{k})\right] (\tau_{1z} + \tau_{2z}) \end{aligned}$$

- ❖ $S_{12}(\mathbf{k}) = 3\, \boldsymbol{\sigma}_1\cdot\mathbf{k}\;\boldsymbol{\sigma}_2\cdot\mathbf{k} - k^2\, \boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2$
- ❖ In the $v_{12}^{\rm S,CD}(\mathbf{k},\mathbf{K})$ only terms up to NLO, involving charge-independence breaking (proportional to T_{12}) and charge-symmetry breaking (proportional to $\tau_{1z}+\tau_{2z}$), are accounted for.
- ❖ $v_{12}^{\rm S}(\mathbf{k},\mathbf{K})$ is regularized via a Gaussian cutoff depending only on the momentum transfer k

$$\widetilde{C}_{R_{\rm S}}(k) = e^{-R_{\rm S}^2 k^2/4} \longrightarrow C_{R_{\rm S}}(r) = \frac{1}{\pi^{3/2} R_{\rm S}^3}\, e^{-(r/R_{\rm S})^2}$$

Momentum-space v_{12}^{S}

- $$\begin{aligned} v_{12}^{\mathrm{S,CI}}(\mathbf{k},\mathbf{K}) \;=\; & \left(C_S + C_1\,k^2 + D_1\,k^4\right) + \left(C_2\,k^2 + D_2\,k^4\right)\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2 + \left(C_T + C_3\,k^2 + D_3\,k^4\right)\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2 \\ & + \left(C_4\,k^2 + D_4\,k^4\right)\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2\,\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2 + \left(C_5 + D_5\,k^2\right)S_{12}(\mathbf{k}) + \left(C_6 + D_6\,k^2\right)S_{12}(\mathbf{k})\,\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2 \\ & + i\left(C_7 + D_7\,k^2\right)\boxed{\mathbf{S}\cdot(\mathbf{K}\times\mathbf{k})} + i\,D_8\,k^2\,\mathbf{S}\cdot(\mathbf{K}\times\mathbf{k})\,\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2 + D_9\left[\mathbf{S}\cdot(\mathbf{K}\times\mathbf{k})\right]^2 + D_{10}\,(\mathbf{K}\times\mathbf{k})^2 \\ & + D_{11}\,(\mathbf{K}\times\mathbf{k})^2\,\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2 + D_{12}\,k^2 K^2 + D_{13}\boxed{k^2 K^2\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2} + D_{14}\,K^2\,S_{12}(\mathbf{k}) \\ & + D_{15}\,K^2\,S_{12}(\mathbf{k})\,\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2 \end{aligned}$$

- $$\begin{aligned} v_{12}^{\mathrm{S,CD}}(\mathbf{k},\mathbf{K}) \;=\; & \left[C_0^{\mathrm{IT}} + C_1^{\mathrm{IT}}\,k^2 + C_2^{\mathrm{IT}}\,k^2\,\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2 + C_3^{\mathrm{IT}}\,S_{12}(\mathbf{k}) + i\,C_4^{\mathrm{IT}}\mathbf{S}\cdot(\mathbf{K}\times\mathbf{k})\right]T_{12} \\ & + \left[C_0^{\mathrm{IV}} + C_1^{\mathrm{IV}}\,k^2 + C_2^{\mathrm{IV}}\,k^2\,\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2 + C_3^{\mathrm{IV}}\,S_{12}(\mathbf{k}) + i\,C_4^{\mathrm{IV}}\mathbf{S}\cdot(\mathbf{K}\times\mathbf{k})\right](\tau_{1z} + \tau_{2z}) \end{aligned}$$

- $S_{12}(\mathbf{k}) = 3\,\boldsymbol{\sigma}_1\cdot\mathbf{k}\;\boldsymbol{\sigma}_2\cdot\mathbf{k} - k^2\,\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2$

- Of course mixed terms as $k^2\,K^2$ or $\mathbf{K}\times\mathbf{k}$ can not Fierz-transformed away

Fierz identities

- As nucleons are fermions, they obey to the Pauli principle and after antysimmetrization the potential become

$$\overline{v} = \frac{1}{2}\left(v - P^{\mathrm{exc}}[v]\right)$$

- The two-nucleon exchange operator assumed to act on the initial nucleons (it can act also on the final nucleons) is defined as

  - $P^{\mathrm{exc}}[v] = \frac{1}{4}(1+\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2)(1+\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2)\,v(\mathbf{k}\to -2\,\mathbf{K}, \mathbf{K}\to -\frac{1}{2}\mathbf{k})$

  - $\mathbf{K} = \frac{1}{2}(\mathbf{p}+\mathbf{p}')$
  - $\mathbf{k} = \mathbf{p}' - \mathbf{p}$

Example Fierz identities: NLO contact potential

- $$\begin{aligned} v^{\mathrm{CT2}}(\mathbf{K},\mathbf{k}) &= \gamma_1\, k^2 + \gamma_2\, k^2\, \boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2 + \gamma_3\, k^2\, \boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2 + \gamma_4\, k^2\, \boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2\, \boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2 + \gamma_5\, K^2 + \gamma_6\, K^2\, \boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2 \\ + &\quad \gamma_7\, K^2\, \boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2 + \gamma_8\, K^2\, \boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2\, \boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2 + i\,\gamma_9\, \mathbf{S}\cdot\mathbf{K}\times\mathbf{k} + i\,\gamma_{10}\, \mathbf{S}\cdot\mathbf{K}\times\mathbf{k}\, \boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2 + \gamma_{11}\, \boldsymbol{\sigma}_1\cdot\mathbf{k}\, \boldsymbol{\sigma}_2\cdot\mathbf{k} \\ &\quad + \gamma_{12}\, \boldsymbol{\sigma}_1\cdot\mathbf{k}\, \boldsymbol{\sigma}_2\cdot\mathbf{k}\, \boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2 + \gamma_{13}\, \boldsymbol{\sigma}_1\cdot\mathbf{K}\, \boldsymbol{\sigma}_2\cdot\mathbf{K} + \gamma_{14}\, \boldsymbol{\sigma}_1\cdot\mathbf{K}\, \boldsymbol{\sigma}_2\cdot\mathbf{K}\, \boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2 \end{aligned}$$

$$\begin{aligned} P^{\,\mathrm{exc}}\, k^2 &= (1+\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2)(1+\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2)\, K^2 \\ P^{\,\mathrm{exc}}\, k^2\, \boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2 &= (3-\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2)(1+\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2)\, K^2 \\ P^{\,\mathrm{exc}}\, k^2\, \boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2 &= (1+\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2)(3-\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2)\, K^2 \\ P^{\,\mathrm{exc}}\, k^2\, \boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2\, \boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2 &= (3-\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2)(3-\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2)\, K^2 \\ P^{\,\mathrm{exc}}\, \boldsymbol{\sigma}_1\cdot\mathbf{k}\, \boldsymbol{\sigma}_2\cdot\mathbf{k} &= (1+\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2)(2\, \boldsymbol{\sigma}_1\cdot\mathbf{K}\, \boldsymbol{\sigma}_2\cdot\mathbf{K} + K^2 - K^2\, \boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2) \\ P^{\,\mathrm{exc}}\, \boldsymbol{\sigma}_1\cdot\mathbf{k}\, \boldsymbol{\sigma}_2\cdot\mathbf{k}\, \boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2 &= (3-\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2)(2\, \boldsymbol{\sigma}_1\cdot\mathbf{K}\, \boldsymbol{\sigma}_2\cdot\mathbf{K} + K^2 - K^2\, \boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2) \\ P^{\,\mathrm{exc}}\, \mathbf{S}\cdot\mathbf{K}\times\mathbf{k} &= -(1+\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2)\, \mathbf{S}\cdot(\mathbf{K}\times\mathbf{k})/2 \\ P^{\,\mathrm{exc}}\, \mathbf{S}\cdot\mathbf{K}\times\mathbf{k}\, \boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2 &= -(3-\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2)\, \mathbf{S}\cdot(\mathbf{K}\times\mathbf{k})/2 \end{aligned}$$

Other relations involving k

- $$\begin{aligned} \overline{v}^{\mathrm{CT2}}(\mathbf{K},\mathbf{k}) &= C_1\, k^2 + C_2\, k^2\, \boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2 + C_3\, k^2\, \boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2 + C_4\, k^2\, \boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2\, \boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2 + i\, C_5\, \mathbf{S}\cdot\mathbf{K}\times\mathbf{k} \\ &\quad + C_6\, \boldsymbol{\sigma}_1\cdot\mathbf{k}\, \boldsymbol{\sigma}_2\cdot\mathbf{k} + C_7\, \boldsymbol{\sigma}_1\cdot\mathbf{k}\, \boldsymbol{\sigma}_2\cdot\mathbf{k}\, \boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2 + (-C_1 - 3C_2 - 3C_3 - 9C_4 - C_6 - 3C_7)\ K^2 \\ &\quad + (-C_1 + C_2 - 3C_3 + 3C_4 + C_6 + 3C_7)\ K^2\, \boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2 + (-C_1 - 3C_2 + C_3 + 3C_4 - C_6 + C_7)\ K^2 \\ &\quad + (-C_1 + C_2 + C_3 - C_4 + C_6 - C_7)\ K^2\, \boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2\, \boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2 + \frac{1}{3} i\, C_5\, \mathbf{S}\cdot\mathbf{K}\times\mathbf{k}\, \boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2 \\ &\quad + (-2C_6 - 6C_7)\ \boldsymbol{\sigma}_1\cdot\mathbf{K}\, \boldsymbol{\sigma}_2\cdot\mathbf{K} + (-2C_6 + 2C_7)\, \boldsymbol{\sigma}_1\cdot\mathbf{K}\, \boldsymbol{\sigma}_2\cdot\mathbf{K}\, \boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2 \end{aligned}$$

- There are only 7 independent couplings at NLO after antisymmetrization and one has the freedom to choose an appropriate base

Solution of the Schrödinger equation with v_{12}: p^2-dependence

- In spin S and isospin T channel, the potential reads (no EM and charge-dependence parts)

$$v_{12}^{TS} = v_{TS}^{c}(r) + v_{T}^{t}(r)\, S_{12} + v_{T}^{b}(r)\, \mathbf{L}\cdot\mathbf{S} + v_{TS}^{q}(r)\, \mathbf{L}^2 \\ + v_{T}^{bb}(r)\, (\mathbf{L}\cdot\mathbf{S})^2 + \left\{ v_{TS}^{p}(r) + v_{T}^{pt}(r)\, S_{12}\, ,\, \mathbf{p}^2 \right\}$$

$$\mathbf{p}^2 = \frac{\mathbf{L}^2}{r^2} - \frac{1}{r}\frac{d^2 r}{dr^2}$$

- Single channels ($J = L$, where L and J are the orbital and total angular momenta), the Schrödinger equation for the reduced radial function $u_{TSJ}(r)$ reads

$$-(1+\overline{v})\, u'' - \overline{v}\,' u' + \left[v - \frac{\overline{v}\,''}{2} - k^2 \right] u = 0$$

$$v_{TSJ} = 2\,\mu \left[v_{TS}^{c} + \delta_{S,1}\left(2\, v_T^{t} - v_T^{b}\right) + J(J+1)\left(v_{TS}^{q} + 2\,\frac{v_{TS}^{p}}{r^2} + \delta_{S,1}\, 4\, \frac{v_T^{pt}}{r^2} \right) + \delta_{S,1} v_T^{bb} \right] + \frac{J(J+1)}{r^2}$$

$$\overline{v}_{TS} = 4\,\mu\left(v_{TS}^{p} + \delta_{S,1}\, v_T^{pt} \right)$$

- $u = \lambda\, w$ ⇨ $2\,(1+\overline{v})\,\lambda' + \overline{v}\,'\lambda = 0$ ⇨ $\lambda = (1+\overline{v})^{-1/2}$

- $w'' = f\, w\;,\qquad (1+\overline{v})\, f = v - \dfrac{(\overline{v}\,'/2)^2}{1+\overline{v}} - k^2$ ⇨ Numerov method

- In coupled channels ($L = J \pm 1 \equiv \pm$) it is convenient to introduce the 2×2 matrices V and $\overline{V}$: the Schrödinger equation can be written as

$$-\left(1+\overline{V}\right) U'' - \overline{V}\,'U' + \left[V - \frac{\overline{V}\,''}{2} - k^2\right] U = 0$$

  - $$v_{--}^{TJ} = 2\,\mu\left[v_{T1}^{c} - 2\,\frac{J-1}{2J+1}\,v_T^t + (J-1)v_T^b + J(J-1)\left(v_{T1}^q + 2\,\frac{v_{T1}^p}{r^2} - 4\,\frac{J-1}{2J+1}\,\frac{v_T^{pt}}{r^2}\right) + (J-1)^2 v_T^{bb}\right] + \frac{J(J-1)}{r^2}$$
  - $$v_{++}^{TJ} = 2\,\mu\left[v_{T1}^{c} - 2\,\frac{J+2}{2J+1}\,v_T^t - (J+2)v_T^b + (J+1)(J+2)\left(v_{T1}^q + 2\,\frac{v_{T1}^p}{r^2} - 4\,\frac{J+2}{2J+1}\,\frac{v_T^{pt}}{r^2}\right) + (J+2)^2 v_T^{bb}\right] + \frac{(J+1)(J+2)}{r^2}$$
  - $$v_{-+}^{TJ} = 12\,\mu\frac{\sqrt{J(J+1)}}{2J+1}\left(v_T^t + 2\,\frac{J^2+J+1}{r^2}\,v_T^{pt}\right), \qquad v_{+-}^{TJ} = v_{-+}^{TJ}$$
  - $$\overline{v}_{--}^{TJ} = 4\,\mu\left(v_{T1}^p - 2\,\frac{J-1}{2J+1}\,v_T^{pt}\right)$$
  - $$\overline{v}_{++}^{TJ} = 4\,\mu\left(v_{T1}^p - 2\,\frac{J+2}{2J+1}\,v_T^{pt}\right)$$
  - $$\overline{v}_{-+}^{TJ} = 24\,\mu\,\frac{\sqrt{J(J+1)}}{2J+1}\,v_T^{pt}\,, \qquad \overline{v}_{+-}^{TJ} = \overline{v}_{-+}^{TJ}$$

- $U = \Lambda\, W$ ⇨ $2\left(1+\overline{V}\right)\Lambda' + \overline{V}\,'\Lambda = 0$ ⇨ Runge-Kutta method
- $W'' = F\,W\,,\qquad \left(1+\overline{V}\right)\Lambda\,F\,\Lambda^{-1} = V - \frac{1}{4}\,\overline{V}\,'(1+\overline{V})^{-1}\,\overline{V}\,' - k^2$ ⇨ Numerov method