Renormalisation group invariance of many-body observables

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TRIUMF Workshop

Progress in ab initio techniques

Outline

Motivations

- Great progress of *ab initio* calculations in nuclei and nuclear matter

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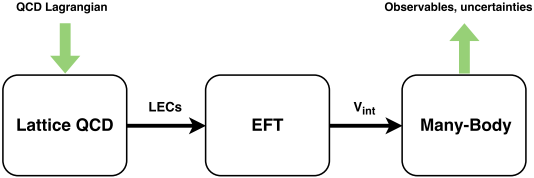
Long term : develop many-body schemes fulfilling EFT requirements

Short term : avoiding cutoff dependence of observables

Interface EFT interaction / Many-body methods

- Traditional view : V_{int} as black box



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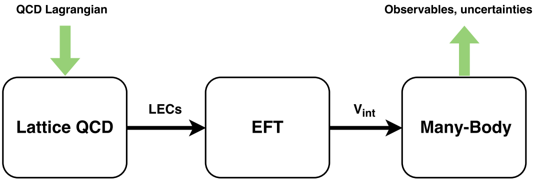
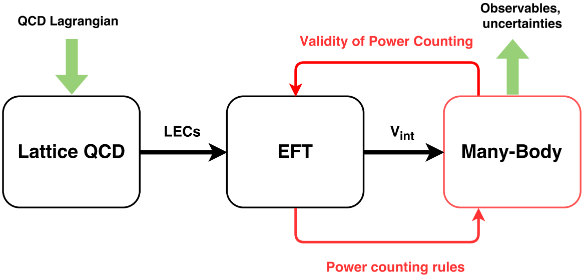


- Adapt many-body scheme to assess proposed power counting



Outline

Modelisation of neutron matter

- Interaction lagrangian

$$\mathcal{L} = N^\dagger \left(i\partial_0 + \frac{\vec{\nabla}^2}{2M} \right) N + C_0(\Lambda) N^\dagger N^\dagger N N$$

- Cutoff regularisation

$$\begin{aligned} V(k, k'; \Lambda) &= C_0(\Lambda)\ v_\Lambda(k) v_\Lambda(k') && \text{Generic} \\ &= C_0(\Lambda)\ \exp -\frac{k^2}{\Lambda^2} \exp -\frac{k'^2}{\Lambda^2} && \text{Gaussian} \end{aligned}$$

- Matching $C_0(\Lambda)$ to the scattering lenght $a_0 = -18.9$ fm

- Many-body schemes considered
  - Hartree-Fock (HF)
  - Particle-particle/hole-hole resummation ladder

  } Analytically & numerically

  - Self Consistent Green's Function (SCGF)

  } Only numerically

Importance of renormalisation consistency

Hartree-Fock : (expanded in $\frac{2\sqrt{2}k_F}{\Lambda}$ for convenience)

$$\frac{E^{HF}}{A}(\Lambda;k_F) = \frac{k_F^2}{2m}\left\{\frac{3}{5} - \frac{mC_0(\Lambda)}{(2\pi)^2}8k_F\left[\frac{1}{12} + O\left(\left(\frac{2\sqrt{2}k_F}{\Lambda}\right)^2\right)\right]\right\}$$

Ladder pp/hh resummation :

$$\frac{E^{Ld}}{A}(\Lambda;k_F) = \frac{k_F^2}{2m}\left\{\frac{3}{5} - \frac{48}{\pi}\int_0^1 \mathrm{d}s\ s^2 \int_0^{\sqrt{1-s^2}} \mathrm{d}\kappa\ \kappa \arctan\left(\frac{k_F I_{\frac{k_F}{\Lambda}}(s,\kappa)}{\frac{4\pi}{mC_0(\Lambda)} + \frac{k_F}{\pi}\tilde{R}_{\frac{k_F}{\Lambda}}(s,\kappa)}\right)\right\}$$

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$$C_0(\Lambda)=\frac{4\pi}{M}a_0$$

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$C_0(\Lambda)$ exactly matched to a_0

$$C_0(\Lambda)=\frac{4\pi}{M}\frac{1}{\frac{1}{a_0}-\frac{\Lambda}{\sqrt{2\pi}}}$$

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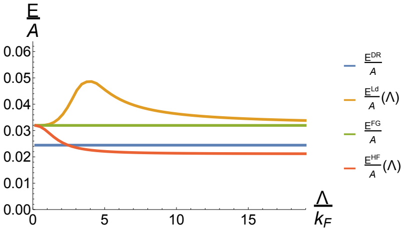

$k_F = 60$ MeV

Ladder pp/hh resummation :

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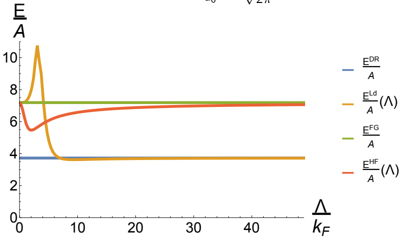
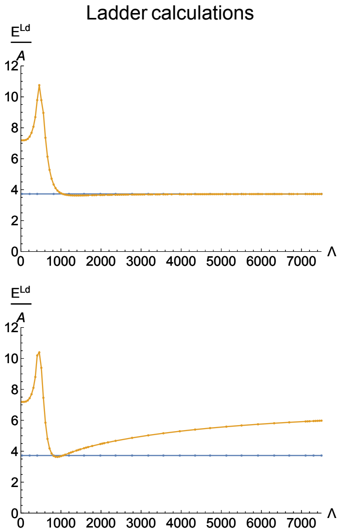


$k_F = 150$ MeV

$\frac{E^{DR}}{A}$ from [Kaiser 11]



How to know if convergence is reached ?

Ideal case :

Analytically proven convergence

$$\lim_{\Lambda\to\infty} \frac{E^{Ld}}{A}(\Lambda; k_F) = \frac{E^{DR}}{A}(k_F) \neq \frac{E^{FG}}{A}(k_F)$$

Real world case :

Slight (e.g. numerical) error on resummation

$$\lim_{\Lambda\to\infty} \frac{E^{Ld}_{app}}{A}(\Lambda; k_F) = \frac{E^{FG}}{A}(k_F)$$

Numerical approximation
should be treated with care

What about self-consistent calculations ?

Outline

General aspects of SCGF

- State-of-the-art SCGF code for nuclear matter

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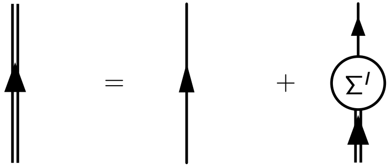
$$G = G^0 + G^0 \Sigma^I G$$



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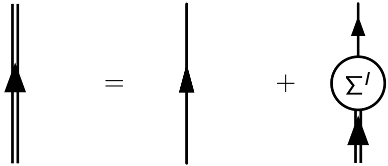
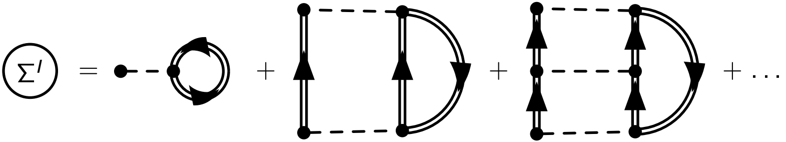


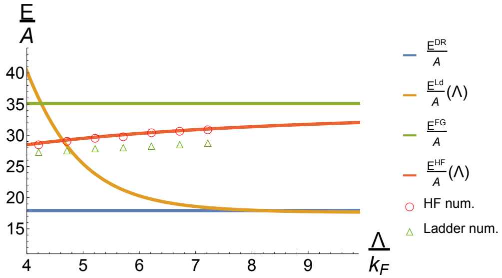
- Ladder ansatz for the self-energy



SCHF $\quad$ 2^{nd} *Born* $\quad$ . . . $\quad$ *Ladder*

Numerical results

- Good reproduction of HF
- Discrepancy at ladder level
  $\rightarrow$ Not adapted code
- Small effects from self-consistency



- Good precision on counter-terms cancellation
- Large range of cutoff variation

Numerically demanding

Outline

Summary and perspectives

▷ Summary

- Many-body methods $\rightarrow$ Consistent with renormalization scheme
- Consistent approaches $\rightarrow$ Validity of power counting ?
- Analytical/numerical analysis

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- Many-body methods → Consistent with renormalization scheme
- Consistent approaches → Validity of power counting ?
- Analytical/numerical analysis
  - → Many-body approximations can lead to cutoff-independent observables
  - → Better understanding of the apparent failure of Ladder approximation
  - → On-going control of critical numerical errors

Summary and perspectives

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- Many-body methods $\rightarrow$ Consistent with renormalization scheme
- Consistent approaches $\rightarrow$ Validity of power counting ?
- Analytical/numerical analysis
  - $\rightarrow$ Many-body approximations can lead to cutoff-independent observables
  - $\rightarrow$ Better understanding of the apparent failure of Ladder approximation
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▷ Perspectives

- Extend resummation while keeping observables $\perp\!\!\!\perp$ regularisation
  - $\rightarrow$ Does this will bring important modifications of observables ?
- Possible to find further truncation scheme ?
- Extend study to symmetric matter/NLOs
- What about error estimations ?

Thank you !

Advisor : V. Somá

Co-advisor : T. Duguet

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J. Yang

Back-up slides

Choose a split of the hamiltonian $H \equiv H_0 + V$ and consider (un)correlated states, related by the Lippmann-Schwinger equation

$$\begin{aligned} H_0|\phi\rangle &\equiv E|\phi\rangle \\ H|\psi\rangle &\equiv E|\psi\rangle \\ V|\psi\rangle &\equiv T|\phi\rangle \end{aligned} \quad \Longrightarrow \quad \begin{aligned} |\psi\rangle &= |\phi\rangle + \frac{1}{E - H_0} V|\psi\rangle \\ T &= V + \frac{1}{E - H_0} T \end{aligned}$$

Thus we get the perturbation expansion

$$T = V + V\frac{1}{E - H_0}V + V\frac{1}{E - H_0}V\frac{1}{E - H_0}V + \dots$$

Main features

$\rightarrow$ Choice of $H_0 \sim$ From what state to start the expansion

$\rightarrow$ Order of expansion

- Time-dependent perturbation expansion of one-body propagator

$$iG_{\alpha\beta}(t,t') = \sum_{m=0}^{+\infty} \frac{(-i)^m}{m!} \int_{-\infty}^{+\infty} \mathrm{d}t_1 \int_{-\infty}^{+\infty} \mathrm{d}t_2 \ldots \int_{-\infty}^{+\infty} \mathrm{d}t_m$$
$$\langle \Phi_0^N | \mathcal{T} \left[V(t_1) V(t_2) \ldots V(t_m) a_\alpha(t) a_\beta^\dagger(t') \right] | \Phi_0^N \rangle$$

- Wick theorem $\rightarrow$ express G in terms of

$$iG^0_{\alpha\beta}(t,t') \equiv \frac{\langle \Phi_0^N | \mathcal{T}[a_\alpha(t) a_\beta^\dagger(t')] | \Phi_0^N \rangle}{\langle \Phi_0^N | \Phi_0^N \rangle}$$
$$iG^0_{\alpha\beta}(\omega) = \frac{\delta_{\alpha a}\delta_{\beta b}}{\omega - e_a + i\epsilon} + \frac{\delta_{\alpha i}\delta_{\beta j}}{\omega - e_i - i\epsilon}$$

where e_a and e_i are single particle/hole energies

Observables from Green's function

- One-body observables (generalisable to *n*-body)

$$\frac{\langle\Psi_0^N|\mathcal{O}|\Psi_0^N\rangle}{\langle\Psi_0^N|\Psi_0^N\rangle} = \sum_{\alpha\beta} \mathcal{O}_{\alpha\beta} \frac{\langle\Psi_0^N|a_\alpha^\dagger a_\beta|\Psi_0^N\rangle}{\langle\Psi_0^N|\Psi_0^N\rangle}$$
$$= \sum_{\alpha\beta} \int_{C\uparrow} \frac{\mathrm{d}\omega}{2\pi i} \; \mathcal{O}_{\alpha\beta} G_{\beta\alpha}(\omega)$$

- Particular case : total energy by GMK sum rules (2-body interactions)

$$\frac{E}{A} = \frac{1}{2} \sum_{\alpha\beta} \int_{C\uparrow} \frac{\mathrm{d}\omega}{2\pi i} \; (t_{\alpha\beta} + \omega\delta_{\alpha\beta})\, G_{\beta\alpha}(\omega)$$

whith $T = \sum_{\alpha\beta} t_{\alpha\beta} a_\alpha^\dagger a_\beta$ the kinetic energy

In-medium propagator

- Unperturbed propagator

$$iG^0_{\alpha\beta}(t,t') \equiv \frac{\langle \Phi_0 | \mathcal{T}[a_\alpha(t) a^\dagger_\beta(t')] | \Phi_0 \rangle}{\langle \Phi_0 | \Phi_0 \rangle}$$

- Particle/hole representation

$$iG^0_{\alpha\beta}(p,\omega) = \delta_{\alpha\beta} \left[\frac{\theta(p-k_f)}{\omega - E_p + i\epsilon} + \frac{\theta(k_f - p)}{\omega - E_p - i\epsilon} \right]$$

- Rewrite it in vacuum/in-medium representation

$$iG^0_{\alpha\beta}(p,\omega) = \delta_{\alpha\beta} \left[\frac{1}{\omega - E_p + i\epsilon} + 2i\pi\delta(\omega - E_p)\theta(k_f - p) \right]$$

Analytical benchmarks

- Hartree-Fock (Erf functions expanded for $\Lambda \gg k_f$)

$$\frac{E^{HF}}{A}(\Lambda;k_f) = \frac{k_f^2}{2m}\left\{\frac{3}{5} - \frac{mC_0(\Lambda)}{(2\pi)^2} 8k_f \left[\frac{1}{12} + O\left(\left(\frac{2\sqrt{2}k_f}{\Lambda}\right)^2\right)\right]\right\}$$

- 2^{nd} Born approximation

$$\frac{E^{2B}}{A}(\Lambda;k_f) = \frac{3}{5}\frac{k_f^2}{2m} - \frac{6C_0(\Lambda)k_f^3}{\pi^2}\int_0^1 \mathrm{d}s\ s^2 \int_0^{\sqrt{1-s^2}} \mathrm{d}\kappa\ \kappa\ I_0(s,\kappa)\ v_\Lambda^2(2\kappa\ k_f)$$
$$\times\left[1 - \frac{mC_0(\Lambda)}{(2\pi)^2}\ k_f \tilde{R}_{\frac{kf}{\Lambda}}(s,\kappa)\right]$$

- Ladder pp/hh resummation with cutoff regularisation

$$\frac{E^{Ld}}{A}(\Lambda;k_f) = \frac{k_f^2}{2m}\left\{\frac{3}{5} - \frac{48}{\pi}\int_0^1 \mathrm{d}s\ s^2 \int_0^{\sqrt{1-s^2}} \mathrm{d}\kappa\ \kappa \arctan\left(\frac{k_f I_{\frac{k_f}{\Lambda}}(s,\kappa)}{\frac{4\pi}{mC_0(\Lambda)} + \frac{k_f}{\pi}\tilde{R}_{\frac{kf}{\Lambda}}(s,\kappa)}\right)\right\}$$

Analytical convergences

- Hartree-Fock

$$\frac{E^{HF}}{A}(\Lambda; k_f) \to \frac{3}{5}\frac{k_f^2}{2m} = \frac{E^{FG}}{A}(k_f)$$

- 2^{nd} Born approximation

$$\frac{E^{2B}}{A}(\Lambda; k_f) \to \frac{E^{FG}}{A}(k_f)$$

- Ladder pp/hh resummation with cutoff regularisation

$$\frac{E^{Ld}}{A}(\Lambda; k_f) \to \frac{k_f^2}{2m}\left\{\frac{3}{5} - \frac{48}{\pi}\int_0^1 \mathrm{d}s\ s^2 \int_0^{\sqrt{1-s^2}} \mathrm{d}\kappa\ \kappa \arctan\left(\frac{k_f I_0(s,\kappa)}{-\frac{1}{a_0} + \frac{kf}{\pi}R_0(s,\kappa)}\right)\right\}$$
$$\neq \frac{E^{FG}}{A}(k_f)$$

Dyson's equation

- Define self-energy from equation of motion of G (2-body interactions)

$$\left(i\partial_{t_1'} + \frac{\nabla_1'^2}{2m}\right) G(1,1') = \delta(1,1') - i \int \mathrm{d}r_2 \; V(r_1 - r_2) G_2(1, r_2, t_1; 1', r_2, t_1^+)$$

$$\int_C \mathrm{d}3 \; \Sigma(1,3) G(3,1') \equiv -i \int \mathrm{d}r_2 \; V(r_1 - r_2) G_2(1, r_2, t_1; 1', r_2, t_1^+)$$

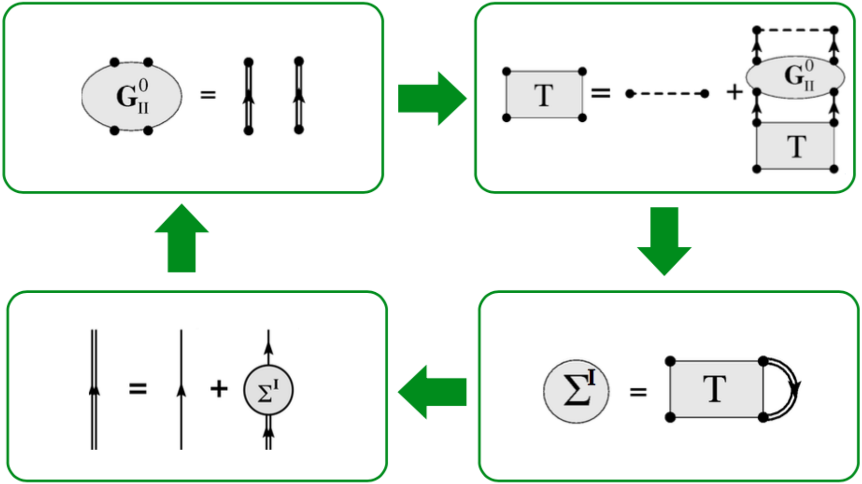
- Hence the Dyson's equation

$$G_{\alpha\beta}(\omega) = G^0_{\alpha\beta}(\omega) + \sum_{\gamma\delta} G^0_{\alpha\gamma}(\omega) \Sigma^I_{\gamma\delta}(\omega) G^0_{\delta\beta}(\omega) + \dots$$

$$G_{\alpha\beta}(\omega) = G^0_{\alpha\beta}(\omega) + \sum_{\gamma\delta} G^0_{\alpha\gamma}(\omega) \Sigma^I_{\gamma\delta}(\omega) G_{\delta\beta}(\omega)$$

where Σ^I is the irreducible self-energy.

Implementation of self-consistent ladder calculations



Brief history of nuclear interactions

- 1935 : Yukawa potential
  - $\rightarrow$ Birth of Meson Theory
- 40s -50s : Pions Theories
  - $\rightarrow$ Troubles with multi-pions, anti-nucleons diagrams, ...
- 60s -00s : One-Boson-Exchange Model $(\rho, \sigma, \omega, \dots)$
  - $\rightarrow$ Good data fitting
  - $\rightarrow$ But no systematic

No renormalization group invariance

- 90s - today : EFT based interactions [S.Weinberg 90 91]
  - $\rightarrow$ *Pragmatic* view : Break RG invariance + estimate error
    [Entem, Machleidt 03] [Epelbaum, Glöckle, Meissner 05]
  - $\rightarrow$ *Canonical* view : Modify power counting and adapt Many-body methods
    [Kaplan, Savage, Wise 96] [Nogga, Timmermans, van Kolck 05]

Motivations for interaction based on EFTs

Folk Theorem

The quantum field theory generated by the most general Lagrangian with some assumed symmetries will produce the most general S matrix incorporating quantum mechanics, Lorentz invariance, unitarity, cluster decomposition and the assumed symmetries. [S.Weinberg 79]

Inputs

- ▷ Degrees of Freedom
  - Nucleons
  - Pions
  - ...
- ▷ Symmetries
  - Space-time
  - Internal
- ▷ LECs $\rightarrow$ Fit to data

Advantages

- Direct connection to underlying theory
- Hierarchy of terms in potential $\rightarrow$ Power counting rules
- Same framework for all A-body interactions
- Estimations of uncertainty and range of validity

How to make an EFT well-defined

Regularization procedure needed

$\rightarrow$ Regulator = modify high-energy physics

$\rightarrow$ Introduce an arbitrary cut-off scale Λ

EFT + Regulator $\equiv$ Well-defined theory

Matching procedure

$\rightarrow$ Compute observables $\mathrm{O}\left(c_i(\Lambda), \Lambda\right)$ at a given order N

$\rightarrow$ Fit LECs to experimental data or to the underlying theory

$\rightarrow$ High-energy physics taken into account in $c_i(\Lambda)$

Renormalization features

- High-energy physics fully included in LECs
- Independence of the observables from the regularization

EFTs for nuclear systems

- Low energy observables in nuclear systems : $Q \ll m_\pi$

  Degrees of freedom : non-relativistic nucleons (+ photons, ...)

$$\rightarrow \quad \mathcal{L}_{\not\pi EFT} = N^\dagger (i\partial_0 + \frac{\vec{\nabla}^2}{2m_N})N + C_0(N^\dagger N)^2 + D_0(N^\dagger N)^3 + \dots$$

- If $Q \sim m_\pi$: D.o.F Nucleons + pions (+ delta + ...)

$$\rightarrow \quad \mathcal{L}_{\chi EFT}$$

Truncation scheme

$\rightarrow$ What diagrams to compute for N^{th} order ?

Expansion of observables

- Separation of scale
  - $\rightarrow$ Low energy observable Q
  - $\rightarrow$ Breakdown scale M
- Expansion of observables $\mathrm{O}_N \propto \left(\frac{Q}{M}\right)^N$
- What diagram gives a contribution of this order ?
- Start with a guess on LECs size then do power counting

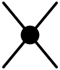
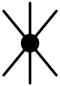
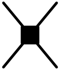
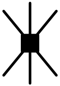
Consistency check

- Convergent observables in negative power of Λ
  - $\rightarrow$ Observable $\perp\!\!\!\perp$ higher energy scale (except through LECs)
- Convergence of observable to experiment when N increases
  - $\rightarrow$ Fail ? Try another guess on LECs
  - $\rightarrow$ No consistent power counting works ? might be wrong D.o.F

$\not{\pi}$*EFT* for Nuclear systems

- Guess on LECs size $\rightarrow$ Proposed power counting for $\not{\pi}$*EFT*

  [Van Kolck 97] [Kaplan, Savage, Wise 98] [Bedaque, Hammer, van Kolck 98 99] . . .

	2-body	3-body	4-body
LO			X
NLO		X	?
N^2*LO*	X		?

Un-natural scattering length a

$\Downarrow$

 $+$  $+ \ldots$ at LO

- Consistency check : LO $A = 4, 6, 16$ & NLO up to $A = 3$

  [Contessi, Lovato, Pederiva, Roggero, Kirsher, van Kolck 17] [Vanasse et al. 13][König et al. 16]

Need to extend consistency check to general A-body observables !

This is where *ab initio* many-body methods enter the game